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Paper IV
Differential Equation

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Paper 104

DIFFERENTIAL EQUATION

UNIT I

Linear second order ordinary differential equations with variable coefficients

1.1 Introduction :

A linear ordinary differential equation of second order is of the form

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \quad --- (1)$$ where P,Q,R are functions of x alone.

It is known that when P and Q are constants, the equation (1) can be solved. In fact, an n^{th} order linear ordinary differential equation , namely,

$$P_0\frac{d^ny}{dx^n} + P_1\frac{d^{n-1}y}{dx^{n-1}} + \cdots\cdots + P_{n-1}\frac{dy}{dx} + P_ny = f(x) \quad --- (2)$$

where f(x) is a function of x but the P_i's, i=0,1,........n are constants can also be solved.

Again a linear homogeneous ordinary differential equation of order n , of the form

$$x^n\frac{d^ny}{dx^n} + P_1x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots\cdots + P_{n-1}\frac{dy}{dx} + P_ny = g(x) \quad --(3)$$

can be integrated . Because, the change of the independent variable x to another variable z by the substitution z =logx renders the equation (3) to an equation of the form (2) with z as independent variable, that is , an equation with constant coefficients. So it can be solved.

However , when P and Q are any functions of x there is no general method for solving an equation of the type (1) . There is even no guarantee that we can solve this equation (1) in closed forms in terms of the elementary functions such as polynomials, trigonometric, hyperbolic, exponential or logarithmic functions in x. Nevertheless, there are certain procedures which at times yield a solution of the equation (1) when the functions P(x) and Q(x) satisfy certain conditions. In what follows, we discuss these procedures towards finding a solution of the equation (1).

1.2 Change of the dependent variable

Consider the second order linear differential equation with variable coefficients,

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R(x) \quad --- (1)$$

For its solution we have to find y as a function of x which satisfies the equation (1) identically. Here y is the dependent variable and x the independent variable.

For this purpose, we put y=uv where u=u(x), v=v(x) where u and v are not yet determined.

We have now

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\text{and}\quad \frac{d^2y}{dx^2} = u\frac{d^2v}{dx^2} + \frac{du}{dx}\frac{dv}{dx} + \frac{dv}{dx}\frac{du}{dx} + v\frac{d^2u}{dx^2}$$

$$= u\frac{d^2v}{dx^2} + 2\frac{du}{dx}\frac{dv}{dx} + v\frac{d^2u}{dx^2}$$

On using these expressions for the derivatives of y, the equation (1) reduces to

$$u\frac{d^2v}{dx^2} + 2\frac{du}{dx}\frac{dv}{dx} + v\frac{d^2u}{dx^2} + P(x)\left[u\frac{dv}{dx} + v\frac{du}{dx}\right] + Q(x)uv = R(x)$$

$$\text{or,}\quad \frac{d^2v}{dx^2} + \left\{\frac{2}{u}\frac{du}{dx} + P(x)\right\}\frac{dv}{dx} + \frac{1}{u}\left\{\frac{d^2u}{dx^2} + P(x)\frac{du}{dx} + Q(x)u\right\}v = \frac{R(x)}{u}$$

$$\text{or,}\quad \frac{d^2v}{dx^2} + P_1(x)\frac{dv}{dx} + Q_1(x)v = R_1(x) \qquad \text{-----}(2)$$

where

$$\left.\begin{aligned} P_1(x) &= \frac{2}{u}\frac{du}{dx} + P(x) \\ Q_1(x) &= \frac{1}{u}\left\{\frac{d^2u}{dx^2} + P(x)\frac{du}{dx} + Q(x)u\right\} \\ \text{and}\quad R_1(x) &= \frac{R(x)}{u} \end{aligned}\right\} \qquad \text{---}(3)$$

Now two cases arise as discussed below : --

Case I: when one integral of the complementary function is known.

Let u(x) be a known part of the complementary function of equation (1) i.e. u is a solution of the equation

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0 \quad \text{- - -}(4)$$

$$\text{So,}\quad \frac{d^2u}{dx^2} + P(x)\frac{du}{dx} + Q(x)u = 0 \qquad \text{i.e.}\qquad Q_1(x) = 0$$

Then, equation (2) reduces to $\dfrac{d^2v}{dx^2} + P_1(x)\dfrac{dv}{dx} = R_1(x)$

On putting $p = \frac{dv}{dx}$, this equation can be reduced to

$$\frac{dp}{dx} + P_1(x)p = R_1(x) \quad --- (5)$$

This is a linear first order differential equation in p and so can be solved.

Its integrating factor is

$$\text{I.F.} = e^{\int P_1(x)dx} = e^{\int\left\{\frac{2}{u}\frac{du}{dx}+P(x)\right\}dx} = e^{2\int\frac{du}{u}} \times e^{\int P(x)dx}$$
$$= e^{2\log u} \times e^{\int p(x)dx} = u^2 e^{\int P(x)dx}$$

Hence the solution of equation (5) is

$$pu^2 e^{\int Pdx} = \int\left[R_1 u^2 e^{\int Pdx}\right]dx + C, \quad \text{C being the arbitrary constant of integration}$$

$$\text{or,} \quad p = \frac{dv}{dx} = \frac{e^{-\int Pdx}}{u^2}\int\left[\frac{R}{u}u^2 e^{\int Pdx}\right]dx + C\frac{e^{-\int Pdx}}{u^2}$$

Integrating this equation further with respect to x , we obtain

$$v = \int\left\{\frac{e^{-\int Pdx}}{u^2}\int\left[Rue^{\int Pdx}\right]dx\right\}dx + C\int\left\{\frac{e^{-\int Pdx}}{u^2}\right\}dx + D,$$ where D is another constant of integration.

So we have found v as a function of x with two arbitrary constants C and D. Having found v we get y=uv involving two arbitrary constants and so it is the complete primitive of the given second order ordinary differential equation (1).

Case II Removal of the first derivative(Reduction to normal form).

If none of u and v is a part of the complementary function of equation (1) , that is, a solution of equation (4), then we choose u such that

$$P_1(x) = \frac{2}{u}\frac{du}{dx} + P(x) = 0 \quad \text{or,} \quad \frac{du}{u} = -\frac{1}{2}P(x)dx \quad \text{or,} \quad u = e^{-\frac{1}{2}\int P(x)dx} \quad ---(6)$$

$$\text{Now} \quad \frac{du}{dx} = \frac{d}{dx}\left\{e^{-\frac{1}{2}\int P(x)dx}\right\} = e^{-\frac{1}{2}\int P(x)dx} \times \frac{d}{dx}\left\{-\frac{1}{2}\int Pdx\right\}$$
$$= -\frac{1}{2}P(x)e^{-\frac{1}{2}\int P(x)dx} = -\frac{1}{2}P(x).u$$

and $\frac{d^2u}{dx^2} = -\frac{1}{2}P(x)\frac{du}{dx} - \frac{1}{2}u\frac{dP(x)}{dx}$

$$\therefore Q_1(x) = Q(x) + \frac{P(x)}{u}\frac{du}{dx} + \frac{1}{u}\frac{d^2u}{dx^2}$$

$$= Q - \frac{1}{2}P(x)u \times \frac{P(x)}{u} + \frac{1}{u}\left[-\frac{1}{2}P(x)\frac{du}{dx} - \frac{1}{2}u\frac{dP(x)}{dx}\right]$$

$$= Q - \frac{1}{2}P^2 - \frac{1}{2}\frac{P}{u}\times\left(-\frac{1}{2}Pu\right) - \frac{1}{2}\frac{dP(x)}{dx}$$

$$= Q - \frac{1}{2}P^2 + \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx}$$

$$\left.\begin{aligned} \text{or,}\quad & Q_1(x) = Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx} \\ \text{and}\quad & R_1(x) = \frac{R(x)}{u} \end{aligned}\right\} \qquad ---(7)$$

On substituting (6), the equation (2) can be reduced to

$$\frac{d^2v}{dx^2} + Q_1(x)v = R_1(x) \quad ---(8) \quad \text{where } Q_1 \text{ and } R_1 \text{ are now given by (7)}$$

Thus by the particular choice of u(x), the first derivative in the equation (2)has been removed. The reduced equation (8) is called the **normal form** of equation (2).

Further, if $Q_1(x) = Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx} = A$, a constant, the equation (2) becomes $\frac{d^2v}{dx^2} + Av = \frac{R}{u}$ which is a linear equation with constant coefficients and so can be solved.

Otherwise, if $Q_1(x) = \frac{A}{x^2}$, the equation (2) reduces to $x^2\frac{d^2v}{dx^2} + Av = R\frac{x^2}{u}$

which is a homogeneous (also called Cauchy equation) equation and can be solved.

The equation (1) is often written as

$$y'' + P(x)y' + Q(x)y = R(x) \quad \text{where } y' = \frac{dy}{dx}, y'' = \frac{d^2y}{dx^2}$$

1.3 Search for a particular integral of

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0 \quad ---(1)$$

Rule I $y=e^{mx}$ is a solution if $m^2+Pm+Q=0$

If $y = e^{mx}$, then $\frac{dy}{dx} = me^{mx}, \frac{d^2y}{dx^2} = m^2e^{mx}$

Thus if $y=e^{mx}$ is a solution, then

$$m^2 e^{mx} + Pme^{mx} + Qe^{mx}=0 \quad \text{or,} \quad m^2 + Pm + Q =0$$

In particular (i) $y=e^x$ will be a solution if $1+ P + Q = 0$

and (ii) $y=e^{-x}$ will be a solution if $1 - P + Q = 0$

Rule II : $y=x^m$ is a solution if $m(m-1)+Pmx+Qx^2=0$

If $y=x^m$, $\frac{dy}{dx} = mx^{m-1}, \frac{d^2y}{dx^2} = m(m-1)x^{m-2}$

Hence if $y=x^m$ is a solution, then $m(m-1)x^{m-2}+Pmx^{m-1}+Qx^m=0$ or, $m(m-1)+Pmx+Qx^2=0$

In particular (i) $y=x$ will be a solution if $P.1.x + Qx^2 =0$ or $P+Qx=0$

and (ii) $y=x^2$ will be a solution if $2+2Px+Qx^2=0$

and (iii) $y=1/x$ is a solution if $-1.(-2) - Px+Qx^2 =0$ i.e. $2-Px+Qx^2=0$

In course of the following examples, we shall find some more functions which are particular solutions of equation (1) under certain conditions to be satisfied by P and Q.

1.4 Solved examples.

Example 1. Solve $\frac{d^2y}{dx^2} - \frac{3}{x}\frac{dy}{dx} + \frac{3}{x^2}y = 2x - 1 \quad --- (I)$

Solution: Here $P(x) = -\frac{3}{x}, Q(x) = \frac{3}{x^2}, R(x) = 2x - 1$

We see that $P + Qx = -\frac{3}{x} + \frac{3}{x^2}.x = 0$

So $y=x$ is a part of the complementary function (C.F.) of the given equation.

We take $y = uv = xv$, where $u =x$ so that

$$\frac{dy}{dx} = 1.v + x\frac{dv}{dx}, \qquad \frac{d^2y}{dx^2} = \frac{dv}{dx} + 1.\frac{dv}{dx} + x\frac{d^2v}{dx^2} = 2\frac{dv}{dx} + x\frac{d^2v}{dx^2}$$

then the given equation becomes

$$2\frac{dv}{dx}+x\frac{d^2v}{dx^2}-\frac{3}{x}\left[v+x\frac{dv}{dx}\right]+\frac{3}{x^2}.xv=2x-1$$

or, $$x\frac{d^2v}{dx^2}+[2-3]\frac{dv}{dx}-\frac{3}{x}v+\frac{3}{x}v=2x-1$$

or, $$\frac{d^2v}{dx^2}-\frac{1}{x}\frac{dv}{dx}=2-\frac{1}{x}$$

On putting , $\frac{dv}{dx} = p$, the last equation becomes $\frac{dp}{dx}-\frac{1}{x}p = 2-\frac{1}{x}$ --- (3)

Its I.F.$=e^{-\int\frac{1}{x}dx} = e^{-\log x} = e^{\log\frac{1}{x}} = \frac{1}{x}$

$\therefore$ solution of equation (3) is $p.\frac{1}{x}=\int\left(2-\frac{1}{x}\right)\frac{1}{x}dx+C$, C is a constant of int. $=2\log x+\frac{1}{x}+C$

or, $p=\frac{dv}{dx} = 2x\log x + 1 + Cx$

Integrating with respect to x we get

$v = \int 2x\log x dx + \int dx + \int Cx dx + D$, D is another constant of integration

$$= 2\left[\log x\int x dx - \int\left\{\frac{1}{x}\int x dx\right\}dx\right] + x + C\frac{x^2}{2} + D$$

$$= 2\left[\frac{x^2}{2}\log x - \int\frac{1}{x}.\frac{x^2}{2}dx\right] + x + C\frac{x^2}{2} + D$$

or, $v = x^2\log x - \frac{x^2}{2} + x + C\frac{x^2}{2} + D$

$= x^2\log x + C_1x^2 + x + D,\qquad C_1 = \frac{(C-1)}{2}$

$\therefore\ y = xv = x^3\log x + x^2 + C_1x^3 + Dx$

which is the required solution.

Example 2 : Solve

$$x\frac{d^2y}{dx^2} - (2x+1)\frac{dy}{dx} + (x+1)y = (x^2+x-1)e^{2x} \quad --- (1)$$

Solution Here $P(x)=-\frac{(2x+1)}{x}$, $Q(x)=\frac{x+1}{x}$, $R(x)=\left(\frac{x^2+x-1}{x}\right)e^{2x}$

and we see that

$$1 + P + Q = 1 - \frac{(2x+1)}{x} + \frac{(x+1)}{x} = 1 - 2 - \frac{1}{x} + 1 + \frac{1}{x} = 0$$

$\therefore y = e^x$ is a part of the C.F.

So, we choose $y = uv = e^x v$, with $u = e^x$

so that $\frac{dy}{dx} = e^x v + e^x \frac{dv}{dx}$, $\frac{d^2y}{dx^2} = e^x(v + \frac{dv}{dx}) + e^x(\frac{dv}{dx} + \frac{d^2v}{dx^2})$

$$= e^x(v + 2\frac{dv}{dx} + \frac{d^2v}{dx^2})$$

Then the given equation reduces to

$\frac{d^2v}{dx^2} + P_1(x)\frac{dv}{dx} = R_1(x)$, where $P_1(x) = \frac{2}{u}\frac{du}{dx} + P(x) = \frac{2}{e^x}.e^x - (2 + \frac{1}{x}) = 2 - 2 - \frac{1}{x} = -\frac{1}{x}$

and $R_1(x) = \frac{R(x)}{u} = \frac{x^2 + x - 1}{x}\frac{e^{2x}}{e^x} = \left(x + 1 - \frac{1}{x}\right)e^x$

or, $\frac{d^2v}{dx^2} - \frac{1}{x}\frac{dv}{dx} = \left(x + 1 - \frac{1}{x}\right)e^x$ on putting $p = \frac{dv}{dx}$, this becomes

$$\frac{dp}{dx} - \frac{1}{x}p = \left(x + 1 - \frac{1}{x}\right)e^x \quad --- (2)$$

Its I.F. $= e^{-\int \frac{1}{x}dx} = e^{-\log x} = e^{\log\frac{1}{x}} = \frac{1}{x}$

$$\therefore \frac{1}{x}p = \int \frac{1}{x}\left(x + 1 - \frac{1}{x}\right)e^x dx + C = \int \left(1 + \frac{1}{x} - \frac{1}{x^2}\right)e^x dx + C$$

$$= \int e^x dx + \int \left(\frac{xe^x - e^x}{x^2}\right)dx + C \quad \because \frac{d}{dx}\left(\frac{e^x}{x}\right) = \frac{e^x}{x} - \frac{e^x}{x^2} = \frac{xe^x - e^x}{x^2}$$

$$= e^x + \int \frac{d}{dx}\left(\frac{e^x}{x}\right)dx + C = e^x + \frac{e^x}{x} + C$$

$$\therefore \quad p = \frac{dv}{dx} = xe^x + e^x + Cx$$

Integrating with respect to x,

$$v = \int xe^x dx + \int e^x dx + \int Cx dx + D = xe^x - \int 1.e^x dx + \int e^x dx + C\frac{x^2}{2} + D$$

$$= xe^x + C_1 x^2 + D$$

$\therefore \quad y = uv = e^x v = C_1 x^2 e^x + De^x + xe^{2x}$ is the required solution.

Example 3 $\frac{d^2y}{dx^2} - 2\tan x \frac{dy}{dx} + 3y = 2\sec x \quad --- (1)$

Solution : Comparing the given equation with

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R(x)$$

we see that $P(x) = -2\tan x$, $\quad Q(x) = 3$, $\quad R(x) = 2\sec x$

Let $u = \sin x$, then $\frac{du}{dx} = \cos x$, $\frac{d^2u}{dx^2} = -\sin x$

Now putting $y = u = \sin x$ in the L.H.S. of equation (1) we get

$$\frac{d^2y}{dx^2} - 2\tan x\frac{dy}{dx} + 3y = -\sin x - 2\tan x.\cos x + 3\sin x = 2\sin x - 2\sin x = 0$$

Thus $y = u = \sin x$ is a particular solution of $\frac{d^2y}{dx^2} - 2\tan x \frac{dy}{dx} + 3y = 0$

For solving the equation (1) we take $\quad y = u(x)v(x) = (\sin x)v(x)$

so that $\frac{dy}{dx} = \frac{dv}{dx}.\sin x + \cos x.v(x)$ and $\frac{d^2y}{dx^2} = \frac{d^2v}{dx^2}\sin x + 2\frac{dv}{dx}\cos x - (\sin x)v(x)$

On substituting these , the equation (1) becomes

$$\frac{d^2v}{dx^2} + P_1(x)\frac{dy}{dx} + Q_1(x)y = R_1(x) \quad --- (2) \quad \text{where}$$

$$P_1(x) = \frac{2}{u}\frac{du}{dx} + P = \frac{2}{\sin x} \times \cos x - 2\tan x = 2(\cot x - \tan x)$$

$$Q_1(x) = 0$$

and $R_1(x) = \frac{R(x)}{u} = \frac{2\sec x}{\sin x} = 2 \times \frac{1}{\sin x \cos x} = \frac{2 \times 2}{\sin 2x} = 4\operatorname{cosec} 2x$

$\therefore$ equation (2) becomes $\quad \frac{d^2v}{dx^2} + 2(\cot x - \tan x)\frac{dv}{dx} = 4\operatorname{cosec} 2x$

Or, on putting $p = \frac{dv}{dx}$, $\frac{dp}{dx} + 2(\cot x - \tan x)p = 4 \operatorname{cosec} 2x$ --- (3)

$$\text{Its I.F} = e^{\int 2(\cot x - \tan x)dx} = e^{2\int \frac{\cos x}{\sin x}dx} \times e^{-2\int \frac{\sin x}{\cos x}dx}$$

$$= e^{\log \sin^2 x} \times e^{\log \cos^2 x} = \sin^2 x \cos^2 x = \frac{1}{4}\sin^2(2x)$$

The solution of equation (3) is

$$p.\frac{1}{4}\sin^2(2x) = \int 4\operatorname{cosec}2x \times \frac{1}{4}\sin^2(2x)dx + C = \int \sin 2x dx + C = -\frac{\cos 2x}{2} + C$$

$$\text{or}, p = \frac{dv}{dx} = -2\cos 2x \times \frac{1}{\sin^2 2x} + 4C\operatorname{cosec}^2 2x$$

$$= -2\cot 2x \times \operatorname{cosec}2x + C_1 \operatorname{cosec}^2 2x. \qquad [C_1 = 4C]$$

Integrating with respect to x,

$$v = -\int 2\cot 2x \times \operatorname{cosec}2x dx + C_1\int \operatorname{cosec}^2 2x dx + C_2$$

$$= \operatorname{cosec}2x - \frac{1}{2}C_1 \cot 2x + C_2$$

$$\therefore y = v\sin x = \frac{\sin x}{2\sin x\cos x} - \frac{1}{2}C_1\frac{\cos 2x}{\sin 2x} \times \sin x + C_2 \sin x$$

$$\because \frac{\cos 2x}{\sin 2x} \times \sin x = \frac{(2\cos^2 x - 1)}{2\sin x \cos s}\sin x$$

$$= \cos x - \frac{1}{2}\sec x$$

or $y = \frac{1}{2}\sec x - \frac{1}{2}C_1\left(\cos x - \frac{1}{2}\sec x\right) + C_2 \sin x$ is the complete solution.

Example 4 Solve $(x + 2)\frac{d^2y}{dx^2} - (2x + 5)\frac{dy}{dx} + 2y = (x + 1)e^x$

Solution : The given equation is

$$\frac{d^2y}{dx^2} - \frac{(2x + 5)}{x + 2}\frac{dy}{dx} + \frac{2}{x + 2}y = \frac{x + 1}{x + 2}e^x \quad --- (1)$$

Here

$P = -\frac{2x + 5}{x + 2}$, $\qquad Q = \frac{2}{x + 2}$, $\qquad R = \frac{x + 1}{x + 2}e^x$

$$= -\frac{2(x + 2) + 1}{x + 2}$$

$$= -2 - \frac{1}{x + 2}$$

Now $2^2 + 2P + Q = 4 - 4 - \frac{2}{x+2} - \frac{2}{x+2} = 0$

$\therefore u = e^{2x}$ is a part of the C.F. of equation (1)

Let the complete solution be $y = uv = e^{2x}v$

Thus the equation (1) can be reduced to

$$\frac{d^2v}{dx^2} + P_1\frac{dv}{dx} + Q_1 v = R_1 \quad ---(2)$$

where $$P_1 = P + \frac{2}{u}\frac{du}{dx} = -\frac{2x+5}{x+2} + \frac{2}{e^{2x}} \times e^{2x} \times 2$$

$$= \frac{-2x-5+4x+8}{x+2} = \frac{2x+3}{x+2}$$

$$Q_1 = 0, \qquad R_1 = \frac{R}{u} = \frac{x+1}{x+2}e^x \times \frac{1}{e^{2x}} = \frac{x+1}{x+2}e^{-x}$$

So equation (2) is

$$\frac{d^2v}{dx^2} + \frac{2x+3}{x+2}\frac{dv}{dx} = \frac{x+1}{x+2}e^{-x} \quad \text{or,} \quad \frac{dp}{dx} + \frac{2x+3}{x+2}p = \frac{x+1}{x+2}e^{-x} \quad ---(3) \text{ where } p = \frac{dv}{dx}$$

Its I.F.= $e^{\int \frac{2x+3}{x+2}dx} = e^{\int\left(2-\frac{1}{x+2}\right)dx} = e^{2x-\log(x+2)} = e^{2x} \times \frac{1}{(x+2)}$

$$\therefore p \times \frac{e2x}{x+2} = \int \frac{(x+1)}{(x+2)}e^{-x} \times \frac{e^{2x}}{x+2}dx + C_1$$

$$= \int \frac{(x+1)}{(x+2)^2}e^x dx + C_1$$

$$\left|\ \frac{(x+1)}{(x+2)^2} = \frac{(x+2)}{(x+2)^2} - \frac{1}{(x+2)^2} = \frac{1}{(x+2)} - \frac{1}{(x+2)^2}\right.$$

$$= \int\left(\frac{e^x}{x+2} - \frac{e^x}{(x+2)^2}\right)dx + C_1$$

or $$p\frac{e^{2x}}{x+2} = \frac{e^{2x}}{x+2} + C_1 \qquad \therefore \frac{dv}{dx} = p = e^{-x} + C_1(x+2)e^{-2x}$$

Integrating with respect to x,

$$v = \int e^{-x}dx + C_1\int (x+2)e^{-2x}dx + C_2$$

$$= -e^{-x} + C_1\left[(x+2)\frac{e^{-2x}}{-2} - \int 1.\frac{e^{-2x}}{-2}dx\right] + C_2$$

$$= -e^{-x} + C_1\left[-\frac{1}{2}(x+2)e^{-2x} + \frac{1}{2}\times\frac{e^{-2x}}{-2}\right] + C_2$$

$$= -e^{-x} - \frac{1}{4}C_1[2x+4+1]e^{-2x} + C_2$$

i.e. $$v = -e^{-x} - \frac{1}{4}C_1 e^{-2x}(2x+5) + C_2$$

$$\therefore y = uv = e^{2x}\left[-e^{-x} - \frac{1}{4}C_1 e^{-2x}(2x+5) + C_2\right]$$

$$= -e^{x} - \frac{1}{4}C_1(2x+5) + C_2 e^{2x}$$ is the required solution.

Example 5 Solve $\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + (x^2+2)y = e^{\frac{1}{2}(x^2+2x)}$ --- (1)

Solution: Here for the given equation

$$P(x) = -2x, \quad Q(x) = x^2 + 2, \quad R(x) = e^{\frac{1}{2}(x^2+2x)}$$

We cannot find an obvious solution $y=u(x)$ of the equation

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + (x^2+2)y = 0$$

that is , there is no obvious part of the C.F. of equation (1)

But $$Q_1 = Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx} = (x^2+2) - \frac{1}{4}(-2x)^2 - \frac{1}{2}(-2)$$
$$= x^2 - 2 - x^2 + 1 = 3$$

To remove the first derivative from the given equation we take $y=uv$

and choose $u = e^{-\frac{1}{2}\int P dx} = e^{-\frac{1}{2}\int -2x dx} = e^{-\frac{x^2}{2}}$

Then the equation (1) will be reduced to

$$\frac{d^2v}{dx^2} + P_1(x)\frac{dv}{dx} + Q_1(x)v = R_1$$

where $P_1(x) = 0,\ Q_1(x) = 3$ and $R_1(x) = \frac{R}{u} = \frac{e^{\frac{1}{2}(x^2+2x)}}{e^{\frac{x^2}{2}}} = e^x$

or, $$\frac{d^2v}{dx^2} + 3v = e^x \qquad ---(2)$$

For C.F. we have $D^2+3=0, D=\frac{d}{dx}$ or, $D=\pm i\sqrt{3}$

$\therefore$ C.F. of equation (2) is $=C_1\cos\sqrt{3}x + C_2\sin\sqrt{3}x$

Particular Integral $P.I.=\frac{1}{D^2+3}e^x = \frac{e^x}{1^2+3} = \frac{e^x}{4}$

$$\therefore v = C_1\cos\sqrt{3}x + C_2\sin\sqrt{3}x + \frac{1}{4}e^x$$

Finally $y=uv=e^{\frac{1}{2}x^2}\left(C_1\cos\sqrt{3}x + C_2\sin\sqrt{3}x + \frac{1}{4}e^x\right)$

Example 6 Solve $\frac{d^2y}{dx^2} + \frac{1}{x^{1/3}}\frac{dy}{dx} + \left(\frac{1}{4x^{2/3}} - \frac{1}{6x^{4/3}} - \frac{6}{x^2}\right)y = 0$ --- (1)

Solution : Here $P = \frac{1}{x^{1/3}}, Q = \frac{1}{4x^{2/3}} - \frac{1}{6x^{4/3}} - \frac{6}{x^2}, R = 0$

Here there is no obvious part of the C.F. of the given equation (1)

$$\text{Now } Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx} = \frac{1}{4x^{2/3}} - \frac{1}{6x^{4/3}} - \frac{6}{x^2} - \frac{1}{4}\left(\frac{1}{x^{1/3}}\right)^2 - \frac{1}{2}\frac{d}{dx}\left(\frac{1}{x^{1/3}}\right)$$
$$= \frac{1}{4x^{2/3}} - \frac{1}{6x^{4/3}} - \frac{6}{x^2} - \frac{1}{4x^{2/3}} - \frac{1}{2}\cdot\frac{-1}{3}\frac{1}{x^{4/3}}$$
$$= -\frac{1}{6x^{4/3}} - \frac{6}{x^2} + \frac{1}{6x^{4/3}} = -\frac{6}{x^2}$$

Then choosing $u=e^{-\frac{1}{2}\int P dx} = e^{-\frac{1}{2}\int x^{-\frac{1}{3}}dx} = e^{-\frac{1}{2}\times\left(\frac{x^{2/3}}{2/3}\right)} = e^{-\frac{3}{4}x^{\frac{2}{3}}}$

And taking $y = uv = e^{-\frac{3}{4}x^{\frac{2}{3}}}v$, the given equation can be reduced to the normal form

$$\frac{d^2v}{dx^2}+\left(Q-\frac{1}{4}P^2-\frac{1}{2}\frac{dP}{dx}\right)v=\frac{R}{u}=0 \quad \text{or,} \quad \frac{d^2v}{dx^2}-\frac{6}{x^2}v=0 \quad \text{or,} \quad x^2\frac{d^2v}{dx^2}-6v=0 \quad --(2)$$

Putting $z=\log x$ or $x=e^z$, We get

$$\frac{dz}{dx}=\frac{1}{x},\frac{dv}{dx}=\frac{dv}{dz}\frac{dz}{dx}=\frac{1}{x}\frac{dv}{dz}$$

$$\frac{d^2v}{dx^2}=\frac{d}{dx}\left(\frac{1}{x}\frac{dv}{dz}\right)=-\frac{1}{x^2}\frac{dv}{dz}+\frac{1}{x}\frac{d^2v}{dz^2}\frac{dz}{dx}=-\frac{1}{x^2}\frac{dv}{dz}+\frac{1}{x^2}\frac{d^2v}{dz^2}$$

$$\text{or,}\quad x^2\frac{d^2v}{dx^2}=\frac{d^2v}{dz^2}-\frac{dv}{dz}=D(D-1)v,\text{where}\quad D=\frac{d}{dz}$$

The equation (2) becomes

$$D(D-1)v-6v=0\quad\text{or,}\quad(D^2-D-6)v=0\quad\text{or,}\quad(D+2)(D-3)v=0$$

$$\therefore v=C_1e^{-2z}+C_2e^{3z}=C_1x^{-2}+C_2x^3$$

Finally $y=uv=e^{-\frac{3}{4}x^{\frac{2}{3}}}(C_1x^{-2}+C_2x^3)$ which is the required solution.

Example 7 Solve $\left(\frac{d^2y}{dx^2}+y\right)\cot x+2\left(\frac{dy}{dx}+y\tan x\right)=\sec x$

Solution: The given equation is

$$\frac{d^2y}{dx^2}+y+2\frac{dy}{dx}\tan x+2y\tan^2 x=\sec x\tan x$$

$$\text{or}\quad\frac{d^2y}{dx^2}+2\frac{dy}{dx}\tan x+(1+2\tan^2 x)y=\sec x\tan x\quad---(1)$$

Here $P=2\tan x$, $Q=1+2\tan^2 x$ and $R=\sec x\tan x$

Also $Q-\frac{1}{4}P^2-\frac{1}{2}\frac{dP}{dx}=(1+2\tan^2 x)-\frac{1}{4}\times 4\tan^2 x-\frac{1}{2}\times 2\sec^2 x$

$=1+\tan^2 x-\sec^2 x=\sec^2 x-\sec^2 x=0$

To remove the first derivative from equation (1), we choose u as

$$u=e^{-\frac{1}{2}\int Pdx}=e^{-\frac{1}{2}\int 2\tan x dx}=e^{\log\cos x}=\cos x$$

Then, on putting $y=uv=(\cos x)v$, the equation (1) can be reduced to the normal form

$$\frac{d^2v}{dx^2}+\left(Q-\frac{1}{4}P^2-\frac{1}{2}\frac{dP}{dx}\right)v=\frac{R}{u}=\frac{\sec x\tan x}{\cos x}=\sec^2 x\tan x$$

$$\text{or,}\ \frac{d^2v}{dx^2}+0=\sec^2 x\tan x$$

Integrating we get $\frac{dv}{dx}=\int\sec^2 x\tan x dx+C_1=\int\tan x d(\tan x)+C_1=\frac{1}{2}\tan^2 x+C_1$

Integrating again, $v = \frac{1}{2}\int \tan^2 x dx + C_1 x + C_2 = \frac{1}{2}\int (\sec^2 x - 1)dx + C_1 x + C2$

$$= \frac{1}{2}[\tan x - x] + C_1 x + C_2 = \frac{1}{2}\tan x + \left(C_1 - \frac{1}{2}\right)x + C_2$$

$\therefore y = uv = \cos x\left(\frac{1}{2}\tan x + C_3 x + C_2\right)$, with $C_3 = \left(C_1 - \frac{1}{2}\right)$ is the required solution.

§ 1.5 Change of the independent variable

In attempting to solve the equation

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R(x) \quad ---(1)$$

if we can find neither a part of its C.F. nor reduce it to a normal form with constant coefficients the methods of solution discussed earlier fail. However, sometimes, on changing the independent variable x to a new variable z the equation (1) may become integrable.

Let x be changed to z by the relation

z=f(x), so that $\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx}, \frac{d^2y}{dx^2} = \frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)^2 + \frac{dy}{dz}\frac{d^2z}{dx^2}$

The equation (1) becomes

$$\frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)^2 + \frac{dy}{dz}\frac{d^2z}{dx^2} + P\frac{dy}{dz}\frac{dz}{dx} + Qy = R$$

or, $$\frac{d^2y}{dz^2} + \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}\frac{dy}{dz} + \frac{Q}{\left(\frac{dz}{dx}\right)^2}y = \frac{R}{\left(\frac{dz}{dx}\right)^2} \quad ---(2)$$

Here we shall encounter the following two cases.

Case I Let z=f(x) be so chosen that the coefficient of $\frac{dy}{dx}$ in equation (2) vanishes. i.e.

$$\frac{d^2z}{dx^2} + P\frac{dz}{dx} = 0 \qquad \text{or,} \qquad \frac{d}{dx}\left(\frac{dz}{dx}\right) + P\frac{dz}{dx} = 0$$

Integrating we get

$$\log\left(\frac{dz}{dx}\right) = -\int Pdx \qquad \therefore \frac{dz}{dx} = e^{-\int Pdx} \text{ or, } z = \int\left[e^{-\int Pdx}\right]dx$$

Then the equation (2) reduces to $\frac{d^2y}{dz^2} + \frac{Q}{\left(\frac{dz}{dx}\right)^2} y = \frac{R}{\left(\frac{dz}{dx}\right)^2}$ --- (3)

This equation can be solved if $\frac{Q}{\left(\frac{dz}{dx}\right)^2}$ is a constant or $\frac{k}{z^2}$, where k is a constant.

Because then equation (3) becomes a linear equation with constant coefficient or another one which can be reduced to such a form.

Case II Let z=f(x) be chosen such that $\frac{dz}{dx} = \sqrt{\frac{\pm Q}{a^2}}$ where the sign with Q is taken so that $\frac{dz}{dx}$ is real and a^2 is a positive constant.

If now $\frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2} = A$, a constant, then equation (2) becomes

$$\frac{d^2y}{dz^2} + A\frac{dy}{dz} + a^2y = \frac{R}{\left(\frac{dz}{dx}\right)^2}$$

which is a linear equation with constant coefficient and so can be solved.

§ 1.6 Solved examples

Example 1 Solve : $\cos x \frac{d^2y}{dx^2} + \sin x \frac{dy}{dx} - 2(\cos^3 x)y = 2\cos^5 x$

Solution : The given equation in standard form is

$$\frac{d^2y}{dx^2} + \tan x \frac{dy}{dx} - 2(\cos^2 x)y = 2\cos^4 x \quad ...... (1)$$

On putting z=f(x), so that

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx}, \frac{d^2y}{dx^2} = \frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)^2 + \frac{dy}{dz}\frac{d^2z}{dx^2}$$

The given equation becomes

$$\frac{d^2y}{dz^2} + \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}\frac{dy}{dz} + \frac{Q}{\left(\frac{dz}{dx}\right)^2} y = \frac{R}{\left(\frac{dz}{dx}\right)^2} \quad --- (2)$$

where P= tanx, Q= $-2\cos^2 x$ and R=$2\cos^4 x$

Let us now choose f(x) such that

$\frac{d^2z}{dx^2} + \tan x \frac{dz}{dx} = 0$ or, $\frac{dp}{dx} + p\tan x = 0$ with $p = \frac{dz}{dx}$. Integrating, we get

$$\log p = -\int \frac{\sin x}{\cos x} dx = \log\cos x \qquad \therefore p = \frac{dz}{dx} = \cos x \text{ and } z = \int \cos x dx = \sin x$$

The equation (2) now becomes

$$\frac{d^2y}{dz^2} - \frac{2\cos^2 x}{\cos^2 x} y = \frac{2\cos^4 x}{\cos^2 x}$$

$$\text{or, } \frac{d^2y}{dz^2} - 2y = 2\cos^2 x = 2(1 - \sin^2 x) = 2(1 - z^2)$$

$$\text{or, } (D^2 - 2)y = 2(1 - z^2), \text{ where } D = \frac{d}{dz}$$

For C.F. we have $D^2-2=0 \Rightarrow D = \pm\sqrt{2}$ $\therefore$ C.F. $= C_1 e^{\sqrt{2}z} + C_2 e^{-\sqrt{2}z}$

$$\text{and P.I.} = \frac{1}{D^2 - 2} \times 2(1 - z^2) = \frac{2}{-2}(1 - \frac{1}{2}D^2)^{-1}(1 - z^2)$$

$$= -(1 + \frac{1}{2}D^2 + ...)(1 - z^2) = -(1 - z^2 + 0 - \frac{1}{2}.2.1) = -(1 - z^2 - 1) = z^2$$

$$\therefore y = C_1 e^{\sqrt{2}z} + C_2 e^{-\sqrt{2}z} + z^2$$

$$\text{i.e. } y = C_1 e^{\sqrt{2}\sin x} + C_2 e^{-\sqrt{2}\sin x} + \sin^2 x$$

which is the complete solution of the given equation .

Example 2 Solve $\frac{d^2y}{dx^2} - \cot x \frac{dy}{dx} - (\sin^2 x)y = \cos x - \cos^3 x$

Solution: Here P= -cotx, Q= $-\sin^2 x$, R=$\cos x - \cos^3 x$

On changing the independent variable from x to z, the given equation becomes

$$\frac{d^2y}{dz^2} + P_1 \frac{dy}{dx} + Q_1 y = R_1 \quad --- (1)$$

where $P_1 = \dfrac{\dfrac{d^2z}{dx^2} + P\dfrac{dz}{dx}}{\left(\dfrac{dz}{dx}\right)^2}, Q_1 = \dfrac{Q}{\left(\dfrac{dz}{dx}\right)^2}, R_1 = \dfrac{R}{\left(\dfrac{dz}{dx}\right)^2}$ a

Let us choose z such that $P_1=0$

$$\text{i.e. } \frac{d^2z}{dx^2} + P\frac{dz}{dx} = 0 \Rightarrow \frac{d}{dx}\left(\frac{dz}{dx}\right) - \cot x \frac{dz}{dx} = 0$$

On integrating

$$\int \frac{\frac{d}{dx}\left(\frac{dz}{dx}\right)}{\frac{dz}{dx}} dx = \int \cot x dx = \int \frac{\cos x}{\sin x} dx \quad \text{or} \quad \log\left(\frac{dz}{dx}\right) = \log \sin x \Rightarrow \frac{dz}{dx} = \sin x$$

$$z = \int \sin x dx = -\cos x$$

This gives $Q_1 = -\dfrac{\sin^2 x}{\sin^2 x} = -1,\quad R_1 = \dfrac{\cos x - \cos^3 x}{\sin^2 x} = \cos x \dfrac{(1-\cos^2 x)}{\sin^2 x} = \cos x = -z$

Then the equation (1) reduces to $\dfrac{d^2y}{dz^2} - y = -z$

For C.F we have $D^2 - 1 = 0$, where $D = \dfrac{d}{dz}$ $\quad \therefore D = \pm 1$,

$$\therefore C.F. = y = C_1e^z + C_2e^{-z} = C_1e^{-\cos x} + C_2e^{\cos x}$$

$$P.I. = \frac{1}{D^2-1}(-z) = (1-D^2)^{-1}(z) = (1+D^2+\cdots)z = z$$

$$\therefore y = C_1e^{-\cos x} + C_2e^{\cos x} - \cos x$$

Example 3 $\dfrac{d^2y}{dx^2} - (1+4e^x)\dfrac{dy}{dx} + 3e^{2x}y = e^{2(x+e^x)}$ $\quad ---(1)$

Solution: Let z be the new independent variable chosen from x so that

$$\frac{dz}{dx} = \sqrt{\frac{Q}{a^2}} = \sqrt{\frac{3e^{2x}}{3}} = e^x, \text{where } a^2 = 3$$

$$\therefore z = e^x$$

This gives $P_1(x) = \dfrac{\dfrac{d^2z}{dx^2} + P\dfrac{dz}{dx}}{\left(\dfrac{dz}{dx}\right)^2} = \dfrac{e^x - (1+4e^x)\times e^x}{e^{2x}} = \dfrac{1-1-4e^x}{e^x} = \dfrac{-4e^x}{e^x} = -4$, a constant

A= - 4

Then the given equation (1) can be reduced to

$$\frac{d^2y}{dz^2} + A\frac{dy}{dz} + a^2y = \frac{R}{\left(\frac{dz}{dx}\right)^2} = \frac{e^{2(x+e^x)}}{(e^x)^2} = \frac{e^{2x}\times e^{2e^x}}{e^{2x}}$$

or $$\frac{d^2y}{dz^2} - 4\frac{dy}{dz} + 3y = e^{2e^x} = e^{2z} \quad ---(2)$$

For C.F. we have $D^2-4D+3=0$ Or, $(D-3)(D-1)=0$ $\therefore D=3,1$. Hence C.F. $=C_1e^{3z}+C_2e^z$

$$P.I. = \frac{1}{(D-3)(D-1)}e^{2z} = \frac{1}{(-1).(+1)}e^{2z} = -e^{2z}$$

$\therefore y = C_1e^{3z} + C_2e^z - e^{2z} = C_1e^{3e^x} + C_2e^{e^x} - e^{2e^x}$ is the required solution.

Example 4 Solve: $\dfrac{d^2y}{dx^2} + \cot x\dfrac{dy}{dx} + 4y\,\text{cosec}^2x = 0 \quad ---(1)$

Solution:

Here $P = \cot x$, $Q=4\text{cosec}^2x$, $R=0$

On changing the independent variable from x to z, the given equation (1) becomes

$$\frac{d^2y}{dz^2} + \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}\frac{dy}{dz} + \frac{Q}{\left(\frac{dz}{dx}\right)^2}y = 0 \quad ---(2)$$

Let us choose z such that

$$\left(\frac{dz}{dx}\right)^2 = \frac{Q}{a^2} = \frac{4\,\text{cosec}^2x}{1} = 4\,\text{cosec}^2x, \qquad a^2 = 1$$

$$\therefore \frac{dz}{dx} = 2\,\text{cosec}\,x, \qquad \frac{d^2z}{dx^2} = -2\,\text{cosec}\,x\cot x$$

and $$z = 2\int \text{cosec}\,x\,dx = 2\log\tan\frac{x}{2}$$

Then the coefficient of $\dfrac{dy}{dz}$ in (2) is

$$P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2} = \frac{-2\cos ecx \cot x + \cot x \times 2\cos ecx}{4\cos ec^2x} = 0$$

and coefficient of y is $Q_1 = \frac{4\cos ec^2x}{4\cos ec^2x} = 1$

Then the equation (2) becomes $\frac{d^2y}{dz^2} + y = 0$

For C.F. we have $D^2+1=0 \Rightarrow D = \pm i$ $\quad \therefore$ C.F. $= A\cos z + B\sin z$

P.I. =0, since the r.h.s. is zero.

Hence the solution of the given equation is

$y = A\cos z + B\sin z = C_1\cos(z + C_2)$, $\quad C_1, C_2$ are arbitrary constants

$$= C_1\cos\left(2\log\tan\frac{x}{2} + C_2\right)$$

Example 5 Solve $x^6\frac{d^2y}{dx^2} + 3x^5\frac{dy}{dx} + a^2y = \frac{1}{x^2}$

Solution : The given equation is

$$\frac{d^2y}{dx^2} + \frac{3}{x}\frac{dy}{dx} + \frac{a^2}{x^6}y = \frac{1}{x^8} \quad ---(1)$$

Here $P = \frac{3}{x}, Q = \frac{a^2}{x^6}$ and $R = \frac{1}{x^8}$

Let us choose z such that $\frac{dz}{dx} = \sqrt{\frac{Q}{k^2}} = \sqrt{\frac{a^2}{x^6} \times \frac{1}{a^2}} = \frac{1}{x^3}$, where $k^2=a^2$

So that $z = -\frac{1}{2x^2}$, and $\frac{d^2z}{dx^2} = -3.\frac{1}{x^4}$

Now $\frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2} = \frac{\frac{-3}{x^4} + \frac{3}{x} \times \frac{1}{x^3}}{\left(\frac{1}{x^3}\right)^2} = 0$

Then, on changing the independent variable from x to z, the equation (1) is reduced to

$$\frac{d^2y}{dz^2}+\frac{Q}{\left(\frac{dz}{dx}\right)^2}y=\frac{R}{\left(\frac{dz}{dx}\right)^2}$$

or, $\frac{d^2y}{dz^2}+\frac{a^2}{x^6}\times x^6y=\frac{1}{x^8}\times x^6=\frac{1}{x^2}=-\frac{1}{2x^2}\times(-2)$

or, $\frac{d^2y}{dz^2}+a^2y=-2z$

or, $(D^2+a^2)y=-2z$

For C.F. we have $D^2+a^2=0\Rightarrow D=\pm ia$ $\therefore$ C.F. $=A\cos az+B\sin az=C_1\cos(az+C_2)$

$$\text{P.I.}=\frac{1}{D^2+a^2}\times(-2z)=-\frac{2}{a^2}\left(1+\frac{D^2}{a^2}\right)^{-1}z=\frac{-2}{a^2}\left(1-\frac{D^2}{a^2}+\ldots\ldots\right)z=-\frac{2}{a^2}z$$

$\therefore$ the solution of the given equation is

$$y=C_1\cos(az+C_2)-\frac{2}{a^2}z$$
$$z=C_1\cos\left(\frac{-a}{2x^2}+C_2\right)+\frac{1}{a^2x^2}$$

§ 1.7 Miscellaneous examples

Example 1 Given that the equation $x(x-1)\frac{d^2y}{dx^2}+\left(\frac{3}{2}-2x\right)\frac{dy}{dx}-\frac{1}{4}y=0$ --- (1)

has a particular integral of the form x^n, prove that $n=-\frac{1}{2}$, and that the primitive of the equation is $y=x^{-\frac{1}{2}}\left[A+B\sin^{-1}\left(x^{1/2}\right)\right]$, where A and B are arbitrary constants.

Solutions: Let $y=x^n$, then $\frac{dy}{dx}=nx^{n-1}$, $\frac{d^2y}{dx^2}=n(n-1)x^{n-2}$

Then the equation (1) yields

$$x(1-x)n(1-n)x^{n-2}+\left(\frac{3}{2}-2x\right)nx^{n-1}-\frac{1}{4}x^n=0$$

or, $n(1-n)\left[x^{n-1}-x^n\right]+n\left[\frac{3}{2}x^{n-1}-2x^n\right]-\frac{1}{4}x^n=0$

$$\text{or,}\quad x^n\left[-n(n-1)-2n-\frac{1}{4}\right]+x^{n-1}\left[n(n-1)+\frac{3}{2}n\right]=0$$

$$\text{or,}\quad -\frac{1}{4}x^n\left[4n^2-4n+8n+1\right]+\frac{1}{2}x^{n-1}\left[2n^2-2n+3n\right]=0$$

$$\text{or,}\quad x^n\left[4n^2+4n+1\right]+x^{n-1}\times(-2)\left[2n^2+n\right]=0\,.$$

$$\text{or,}\quad (2n+1)^2x^n-2n(2n+1)x^{n-1}=0$$

This equation will be satisfied for arbitrary values of x, if we have

$$(2n+1)^2=0 \quad\text{and}\quad -2n(2n+1)=0 \;\Rightarrow n=-\frac{1}{2} \Rightarrow n=0 \text{ or } n=-\frac{1}{2}$$

hence we take $n=-\frac{1}{2}$. Thus $y=x^n=x^{-\frac{1}{2}}$ is a solution of equation (1).

Now to obtain the general solution of equation (1), we take $y=uv=x^{-\frac{1}{2}}v(x)$

so that $\dfrac{dy}{dx}=x^{-\frac{1}{2}}\dfrac{dv}{dx}-\dfrac{1}{2}x^{-\frac{3}{2}}.v$

$$\frac{d^2y}{dx^2}=\frac{d^2v}{dx^2}x^{-\frac{1}{2}}+\frac{dv}{dx}\left(-\frac{1}{2}x^{-\frac{3}{2}}\right)+\frac{dv}{dx}\left(-\frac{1}{2}x^{-\frac{3}{2}}\right)+v\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)x^{-\frac{5}{2}}$$

$$=x^{-\frac{1}{2}}\frac{d^2v}{dx^2}-x^{-\frac{3}{2}}\frac{dv}{dx}+\frac{3}{4}x^{-\frac{5}{2}}v$$

On substituting these, expressions for y, $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$, the equation (1) becomes

$$x(x-1)\left[\frac{d^2v}{dx^2}x^{-\frac{1}{2}}-\frac{dv}{dx}x^{-\frac{3}{2}}+\frac{3}{4}vx^{-\frac{5}{2}}\right]+\left(\frac{3}{2}-2x\right)\left[\frac{dv}{dx}x^{-\frac{1}{2}}-\frac{1}{2}vx^{-\frac{3}{2}}\right]-\frac{1}{4}vx^{-\frac{1}{2}}=0$$

$$\text{or,}\quad \left(x^{\frac{1}{2}}-x^{\frac{3}{2}}\right)\frac{d^2v}{dx^2}+\left[\left(x^{\frac{1}{2}}-x^{-\frac{1}{2}}\right)+\left(\frac{3}{2}x^{-\frac{1}{2}}-2x^{\frac{1}{2}}\right)\right]\frac{dv}{dx}$$

$$+\left[\frac{3}{4}\left(x^{-\frac{3}{2}}-x^{-\frac{1}{2}}\right)-\frac{1}{2}\left(\frac{3}{2}x^{-\frac{3}{2}}-2x^{-\frac{1}{2}}\right)-\frac{1}{4}x^{-\frac{1}{2}}\right]v=0$$

$$\text{or,}\; x^{\frac{1}{2}}(1-x)\frac{d^2v}{dx^2}+\left(\frac{1}{2}x^{-\frac{1}{2}}-x^{\frac{1}{2}}\right)\frac{dv}{dx}=0$$

on multiplying by $x^{\frac{1}{2}}$,

$$x(1-x)\frac{d^2v}{dx^2} + \left(\frac{1}{2} - x\right)\frac{dv}{dx} = 0$$

$$\text{or,}\quad \frac{dp}{dx} + \frac{\frac{1}{2}(1-2x)}{(x-x^2)}p = 0 \qquad \text{where} \qquad p = \frac{dv}{dx}$$

$$\text{or}\quad \int\frac{dp}{p} = -\int\frac{\frac{1}{2}(1-2x)dx}{x-x^2} + \text{cons tan t}$$

$$\text{or}\quad \log p = -\frac{1}{2}\log(x-x^2) + \text{cons tan t}$$

$$= \log\frac{1}{(x-x^2)^{\frac{1}{2}}} + \log A.\ \text{where A is a constant}$$

$$\therefore p = \frac{dv}{dx} = \frac{A}{(x-x^2)^{\frac{1}{2}}}$$

Integrating with respect to x ,

$$v = A\int\frac{dx}{(x-x^2)^{\frac{1}{2}}} + B = A\int\frac{dx}{x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}} + B$$

We put $x^{\frac{1}{2}} = \theta$ so that $\frac{1}{2}x^{-\frac{1}{2}}dx = d\theta$ or $\frac{dx}{x^{\frac{1}{2}}} = 2d\theta$

$$\therefore v = A\int\frac{2d\theta}{\sqrt{1-\theta^2}} + B = 2A\sin^{-1}\theta + B$$

$$\text{or, } v = 2A\sin^{-1}(\sqrt{x}) + B$$

$$\text{Finally } y = x^{-\frac{1}{2}}v = x^{-\frac{1}{2}}\left[2A\sin^{-1}(\sqrt{x}) + B\right]$$

Example 2: Solve $\frac{d^2y}{dx^2} = \sec^2 y \tan y \quad --- (1)$

Solution: Multiplying both sides of the given equation by $2\frac{dy}{dx}$ we get

$$2\frac{dy}{dx}\frac{d^2y}{dx^2} = \sec^2 y\tan y \times 2\frac{dy}{dx}$$

$$\text{or, } \frac{d}{dx}\left(\frac{dy}{dx}\right)^2 = \frac{d}{dx}\left(\tan^2 y\right)$$

Integrating with respect to x, we get

$$\int \frac{d}{dx}\left(\frac{dy}{dx}\right)^2 dx = \int \frac{d}{dx}\left(\tan^2 y\right)dx + C_1$$

$$\text{or, } \int d\left(\frac{dy}{dx}\right)^2 = \int d\left(\tan^2 y\right) + C_1$$

$$\text{or, } \left(\frac{dy}{dx}\right)^2 = \tan^2 y + C_1$$

or, $\frac{dy}{dx} = \sqrt{\tan^2 y + C_1}$, taking positive sign only.

$$\therefore dx = \frac{dy}{\sqrt{\tan^2 y + C_1}} = \frac{dy}{\sqrt{\frac{\sin^2 y}{\cos^2 y} + C_1}} = \frac{\cos y dy}{\sqrt{\sin^2 y + C_1 \cos^2 y}}$$

$$= \frac{\cos y dy}{\sqrt{\sin^2 y + C_1(1 - \sin^2 y)}} = \frac{\cos y dy}{\sqrt{C_1 + (1 - C_1)\sin^2 y}}$$

Putting siny = v, so that cosydy=dv, we get

$$dx = \frac{dv}{\sqrt{C_1 + (1 - C_1)v^2}} = \frac{dv}{\sqrt{C_1 - (C_1 - 1)v^2}}$$

$$= \frac{1}{\sqrt{C_1}} \frac{dv}{\sqrt{1 - \left(\sqrt{\frac{C_1 - 1}{C_1}}v\right)^2}} = \frac{1}{\sqrt{C_1}} \frac{d\left(\sqrt{\frac{C_1 - 1}{C_1}}v\right)}{\sqrt{1 - \left(\sqrt{\frac{C_1 - 1}{C_1}}v\right)^2}} \times \sqrt{\frac{C_1}{C_1 - 1}}$$

On integrating ,

$$x = \frac{1}{\sqrt{C_1 - 1}} \sin^{-1}\left(\sqrt{\frac{C_1 - 1}{C_1}}v\right) + C_2$$

$$\text{or} \quad \sin^{-1}\left(\sqrt{\frac{C_1 - 1}{C_1}} \sin y\right) = \sqrt{C - 1}(x - C_2)$$

$$\text{or, } \sqrt{\frac{C_1 - 1}{C_1}} \sin y = \sin\left\{\sqrt{C_1 - 1}(x - C_2)\right\}$$

$$\therefore \sin y = \sqrt{\frac{C_1}{C_1 - 1}} \sin\left[\sqrt{C_1 - 1}(x - C_2)\right]$$

which gives the solution of the given equation

Example 3 : $\frac{d^2y}{dx^2} = y^3 - y$, given that $\frac{dy}{dx} = 0$ when y=1

Solution: we have

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}(p)=\frac{dp}{dy}\frac{dy}{dx}=p\frac{dp}{dy} \qquad \text{where } p=\frac{dy}{dx}$$

Then the given equation becomes

$$p\frac{dp}{dy} = y^3 - y$$

$$\text{or, } \int p\,dp = \int y^3dy - \int y\,dy + \text{cons tan t}$$

$$\text{or, } \frac{p^2}{2} = \frac{y^4}{4} - \frac{y^2}{2} + \text{cons tan t}$$

$$\text{or, } \left(\frac{dy}{dx}\right)^2 = p^2 = \frac{1}{2}y^4 - y^2 + C \text{ - - - - —} > (2)$$

Given that $\frac{dy}{dx} = 0$, when y=1, $\therefore$ equation (2) gives $0 = \frac{1}{2} - 1 + C$ or $C = \frac{1}{2}$

Thus

$$\left(\frac{dy}{dx}\right)^2 = \frac{1}{2}y^4 - y^2 + \frac{1}{2} = \frac{1}{2}\left(y^4 - 2y^2 + 1\right) = \frac{1}{2}(y^2-1)^2$$

$$\text{or, } \quad \frac{dy}{dx} = \pm\frac{1}{\sqrt{2}}(y^2-1)$$

$$\text{integrating, } \int dx = \pm\sqrt{2}\int\frac{dy}{y^2-1} + \text{cons tan t}$$

$$\text{or, } \frac{1}{\sqrt{2}}x = \pm\int\frac{dy}{(y-1)(y+1)} + \frac{C}{\sqrt{2}}, \text{ C is a constant}$$

$$\text{or } \quad \pm\left(\frac{x-C}{\sqrt{2}}\right) = \frac{1}{2}\int\left(\frac{1}{y-1} - \frac{1}{y+1}\right)dy = \frac{1}{2}\left[\log(y-1) - \log(y+1)\right]$$

$$= \frac{1}{2}\log\left(\frac{y-1}{y+1}\right)$$

$$\text{or, } \log\left(\frac{y-1}{y+1}\right) = \pm\sqrt{2}(x-c) = -\sqrt{2}(x-c), \text{taking only the –ve sign}$$

$$\text{or, } \frac{y-1}{y+1} = \frac{e^{-\sqrt{2}(x-c)}}{1}$$

$$\text{or, } \frac{y-1+y+1}{y-1-y-1} = \frac{e^{-\sqrt{2}(x-c)}+1}{e^{-\sqrt{2}(x-c)}-1}$$

$$\text{or, } \frac{2y}{-2} = \frac{e^{-\sqrt{2}(x-c)} \times e^{\frac{x-c}{\sqrt{2}}} + e^{\frac{x-c}{\sqrt{2}}}}{e^{-\sqrt{2}(x-c)} \times e^{\frac{x-c}{\sqrt{2}}} - e^{\frac{x-c}{\sqrt{2}}}}$$

$$\left| \begin{aligned} & -\sqrt{2}(x-c) + \frac{x-c}{\sqrt{2}} \\ & = (x-c)\left[\frac{1}{\sqrt{2}} - \sqrt{2}\right] \\ & = (x-c)\left[\frac{1-2}{\sqrt{2}}\right] \\ & = -\frac{1}{\sqrt{2}}(x-c) \end{aligned} \right.$$

$$\text{or, } -y = \frac{e^{-\frac{1}{\sqrt{2}}(x-c)} + e^{\frac{1}{\sqrt{2}}(x-c)}}{e^{-\frac{1}{\sqrt{2}}(x-c)} - e^{\frac{1}{\sqrt{2}}(x-c)}} \quad \text{or, } \quad y = \frac{e^{\frac{1}{\sqrt{2}}(x-c)} + e^{-\frac{1}{\sqrt{2}}(x-c)}}{e^{\frac{1}{\sqrt{2}}(x-c)} - e^{-\frac{1}{\sqrt{2}}(x-c)}} = \coth\left(\frac{x-c}{\sqrt{2}}\right)$$

§ 1.8 Method of variation of parameters

The method of variation of parameters is used to find the complete primitive of a linear second order ordinary differential equation when its complementary function is known.

Let the equation be $\dfrac{d^2y}{dx^2} + P(x)\dfrac{dy}{dx} + Q(x)y = R(x)$ --- (1)

and its complementary function (C.F.) be $y = C_1u(x) + C_2v(x)$ -------(2)

where C_1 and C_2 are arbitrary constants. Then u(x) and v(x) are two independent solutions of the equation

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0 \quad --- (3)$$

so that $\dfrac{d^2u}{dx^2} + P\dfrac{du}{dx} + Qu = 0$ and $\dfrac{d^2v}{dx^2} + P\dfrac{dv}{dx} + Q = 0$ --- (4)

Clearly, when $R(x) \neq 0$, the expression $C_1u(x)+C_2v(x)$ will not be the complete primitive of the equation (1).

We now assume that $y=Au+Bv$ ------- (5)

is the complete solution of equation (1), where A and B are no longer constants but functions of x to be so determined that the equation (1) will be satisfied by the expression (5). Thus the constants C_1 and C_2 in the expression (2) which is the solution of equation (3) are replaced by the functions A(x) and B(x) respectively to obtain the solution of the equation (1), For this reason this method is known as the **method of variation of parameters.**

Differentiating (5) with respect to x we get $y_1 = \frac{dy}{dx} = A_1(x)u + B_1(x)v + Au_1 + Bv_1$ ------ (6)

where $A_1 = \frac{dA}{dx}$, $B_1 = \frac{dB}{dx}$, $u_1 = \frac{du}{dx}$ and $v_1 = \frac{dv}{dx}$

We observe that the unknown functions A(x) and B(x) are so far connected by the single equation (5). But to determine them we require two equations (or conditions).

We can now impose on them the condition:- $A_1(x)u + B_1(x)v = 0$ ------- (7)

Then equation (6) reduces to $y_1 = Au_1 + Bv_1$ ---------------------- (8)

Differentiating (8) again we obtain $y_2 = \frac{d^2y}{dx^2} = A_1u_1 + B_1v_1 + Au_2 + Bv_2$ ---(9)

Substituting the expressions for y, y_1, and y_2 from (5), (8) and (9) in equation (1) we obtain

$$[Au_2 + Bv_2 + A_1u_1 + B_1v_1] + P[Au_1 + Bv_1] + Q(Au + Bv) = R$$

or, $$A(u_2 + Pu_1 + Qu) + B(v_2 + Pv_1 + Qv) + (A_1u_1 + B_1v_1) = R$$

The expressions within the first two brackets vanish by virtue of (4)

$$\therefore A_1u_1 + B_1v_1 = R \quad --- (10)$$

Solving equation (7) and (10) for A_1 and B_1 we get

$$A_1 = \frac{dA}{dx} = -\frac{vR}{uv_1 - u_1v} \quad \& \quad B_1 = \frac{dB}{dx} = \frac{uR}{uv_1 - u_1v}$$

integrating these, $A(x) = f(x) + k_1$ and $B = g(x) + k_2$,where

$f(x) = \int \frac{vRdx}{u_1v - uv_1}$ and $g(x) = \int \frac{uRdx}{uv_1 - u_1v}$ where k_1 and k_2 are two constants of integration .

Putting these expressions for A and B in (5) we get the required solutions as

$$y = (f(x) + k_1)u + (g(x) + k_2)v$$

Note : The method of variation of parameter can be applied only when the complete C.F. of the given equation is known. It is applied when the complete C.F. is known, but it is difficult to obtain the particular integral of an equation .

§ 1.9 Solved examples

Example1. Solve the equation $\frac{d^2y}{dx^2} + n^2y = \sec nx$ --- (1)

Solution : The C.F. on solution of $\frac{d^2y}{dx^2} + n^2y = 0$ is $y = C_1\cos nx + C_2\sin nx$, where C_1 and C_2 are constants.

Let us assume the solution of equation (1) as $y = A\cos nx + B\sin nx$ ------- (2)
where A and B are functions of x to be determined.

From (2) we get $\frac{dy}{dx} = -An\sin nx + Bn\cos nx + A_1\cos nx + B_1\sin nx$

The functions A(x) and B(x) are so chosen as to satisfy the condition

$$A_1\cos nx + B_1\sin nx = 0 \quad --- (3)$$

Then $\frac{dy}{dx} = -An\sin nx + Bn\cos nx$

and

$$\frac{d^2y}{dx^2} = -An^2\cos nx - Bn^2\sin nx - A_1n\sin nx + B_1n\cos nx$$
$$= -n^2(A\cos nx + B\sin nx) - A_1n\sin nx + B_1n\cos nx$$
$$= -n^2y - A_1n\sin nx + B_1n\cos nx$$

Putting the expressions for y, $\frac{dy}{dx}$, and $\frac{d^2y}{dx^2}$ in (1) we get

$$\left[-n^2y - A_1n\sin nx + B_1n\cos nx\right] + n^2y = \sec nx$$
$$\text{or, } -A_1n\sin nx + B_1n\cos nx = \sec nx \quad --- (4)$$

Multiplying (3) by nsinnx and (4) by cosnx we get

$$-A_1n\cos nx.\sin nx + B_1n\sin^2 nx = 0$$
$$-A_1n\cos nx.\sin nx + B_1n\cos^2 nx = \sec nx\cos nx$$

Adding the last two equations, one gets

$$nB_1(\sin^2 nx + \cos^2 nx) = 1 \qquad \therefore B_1 = \frac{1}{n}$$

Then, $A_1 = -B_1 \frac{\sin nx}{\cos nx} = -\frac{1}{n}\tan nx$

$$\therefore A = \int \frac{dA}{dx}dx = \int -\frac{1}{n}\tan nx dx + C_1$$

$$= -\frac{1}{n}\int \frac{\sin nx}{\cos nx}dx + C_1 = \frac{1}{n}\int \frac{-\sin nx}{\cos nx}d(nx).\frac{1}{n} + C_1 = \frac{1}{n^2}\log\cos nx + C_1$$

and $B = \int B_1 dx = \frac{1}{n}\int dx = \frac{1}{n}x + C_2$

$\therefore$ the complete solution is $y = A\cos nx + B\sin nx$

or, $y = \left(\frac{1}{n^2}\log\cos nx + C_1\right)\cos nx + \left(\frac{x}{n} + C_2\right)\sin nx$

$$= C_1\cos nx + C_2\sin nx + \frac{\cos nx}{n^2}\log\cos nx + \frac{x}{n}\sin nx.$$

Example 2 Solve the equation $\frac{d^2y}{dx^2} + 4y = 4\tan 2x$ --- (1)

Solution : The C.F. on solution of $\frac{d^2y}{dx^2} + 4y = 0$ is

$y = C_1\cos 2x + C_2\sin 2x$, where C_1 and C_2 are constants.

Let us assume the solution of equation (1) as $y = A\cos 2x + B\sin 2x$ ------ (2)

where A and B are functions of x to be determined.

From (2) we get $\frac{dy}{dx} = -2A\sin 2x + 2B\cos 2x + A_1\cos 2x + B_1\sin 2x$

We impose the following additional condition on A(x) and B(x):

$A_1\cos 2x + B_1\sin 2x = 0$ --- (3)

Then $\frac{dy}{dx} = -2A\sin 2x + 2B\cos 2x$

and

$$\frac{d^2y}{dx^2} = -4A\cos 2x - 4B\sin 2x - 2A_1\sin 2x + 2B_1\cos 2x$$
$$= -4(A\cos 2x + B\sin 2x) - 2A_1\sin 2x + 2B_1\cos 2x$$
$$= -4y - 2A_1\sin 2x + 2B_1\cos 2x$$

Putting the expressions for y, $\frac{dy}{dx}$, and $\frac{d^2y}{dx^2}$ in (1) we get

$$[-4y - 2A_1 \sin 2x + 2B_1 \cos 2x] + 4y = 4 \tan 2x$$
$$\text{or,} -2A_1 \sin 2x + 2B_1 \cos 2x = -2 \tan 2x \quad --- (4)$$

Solving equation (3) and (4) for A_1 and B_1, we get

$$A_1 = -2 \tan 2x \times \sin 2x,$$
$$B_1 = -A_1 \frac{\cos 2x}{\sin 2x} = +2 \tan 2x \times \sin 2x \times \cot 2x = 2 \sin 2x$$

$$\therefore A = \int A_1 dx = -2\int \tan 2x. \sin 2x dx + C_1$$
$$= -\tan 2x \int \sin 2x dx - \int \left(2 \sec^2 2x \int \sin 2x dx\right) dx + C_1$$
$$= \tan 2x. \frac{\cos 2x}{2} - 2\int 2 \sec^2 2x \frac{\cos 2x}{2} dx + C_1$$
$$= \sin 2x - 2\int \sec 2x dx + C_1$$
$$= \sin 2x - \log(\sec 2x + \tan 2x) + C_1$$

and

$$B = \int B_1 dx = 2\int \sin 2x dx + C_2 = -\cos 2x + C_2$$

$\therefore$ the complete solution is

$$y = A\cos 2x + B\sin 2x$$
$$= C_1 \cos 2x + C_2 \sin 2x - \log(\sec 2x + \tan 2x). \cos 2x.$$

Example 3 Solve the equation

$$x^2 \frac{d^2y}{dx^2} + 3x \frac{dy}{dx} + y = \frac{1}{(1-x)^2} \quad --- (1)$$

Solution : To obtain the C.F. of equation , we have to solve the equation

$$x^2 \frac{d^2y}{dx^2} + 3x \frac{dy}{dx} + y = 0 \quad --- (2)$$

Let us put $x = e^z$ or $\log x = z$,

$$\text{So that } \frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz} \quad \text{or, } x\frac{dy}{dx} = \frac{dy}{dz}$$

$$\text{and} \quad x\frac{d^2y}{dx^2} + \frac{dy}{dx} = \frac{d^2y}{dz^2}\frac{dz}{dx} = \frac{1}{x}\frac{d^2y}{dz^2}$$

or $\quad x^2\frac{d^2y}{dx^2}=\frac{d^2y}{dz^2}-x\frac{dy}{dx}=\frac{d^2y}{dz^2}-\frac{dy}{dz}$

Then equation (2) becomes

$$\frac{d^2y}{dz^2}-\frac{dy}{dz}+3\frac{dy}{dz}+y=0$$

$$(D^2+2D+1)y=0 \text{ where } D=\frac{d}{dz}$$

$$\text{or, } (D+1)^2y=0,$$

Auxiliary equation is $(D+1)^2=0$ or, $D=-1,-1$

$$\therefore y=(C_1z+C_2)e^{-z}=(C_1\log x+C_2)\frac{1}{x}=C_1\frac{\log x}{x}+C_2\frac{1}{x}$$

or, $\quad y=C_1u+C_2v$

where $u=\frac{\log x}{x}$, $v=\frac{1}{x}$ and C_1,C_2 are constants.

To solve equation (1), we take as the complete primitive

$Y=A(x)u+B(x)v$ ------ (3) where A and B are to be determined.

Then $y_1=\frac{dy}{dx}=A_1u+B_1v+Au_1+Bv_1$

We now impose on A and B, the condition

$$A_1u+B_1v=0 \quad --- (4)$$

$$\therefore y_1=Au_1+Bv_1 \quad --- (5)$$

and $y_2=\frac{d^2y}{dx^2}=A_1u_1+B_1v_1+Au_2+Bv_2$ --- (6)

Substituting these expressions for y, y_1 and y_2 in equation (1), we get

$$x^2(Au_2+Bv_2+A_1u_1+B_1v_1)+3x(Au_1+Bv_1)+(Au+Bv)=\frac{1}{(1-x)^2}$$

$$\text{or, } A(x^2u_2+3xu_1+u)+B(x^2v_2+3xv_1+v)+x^2(A_1u_1+B_1v_1)=\frac{1}{(1-x)^2}$$

Here the coefficients of A and B vanish since u and v are the solutions of the equation (2)

$\therefore$ the last equation becomes

$$A_1u_1 + B_1v_1 = \frac{1}{x^2(1-x)^2} = R(\text{say}) --- (7)$$

Also, we already have $A_1u+B_1v=0$ ------- (4)

Solving these two equations for A_1 and B_1 we get

$$A_1 = \frac{Rv}{vu_1 - uv_1}, B_1 = -\frac{Ru}{vu_1 - uv_1} \text{ where } u_1 = \frac{d}{dx}\left(\frac{\log x}{x}\right) = \frac{1}{x^2} - \frac{\log x}{x^2}, v_1 = -\frac{1}{x^2}$$

$$\therefore vu_1 - uv_1 = \frac{1}{x}\left(\frac{1}{x^2} - \frac{\log x}{x^2}\right) - \frac{\log x}{x}\left(\frac{-1}{x^2}\right) = \frac{1}{x^3} - \frac{\log x}{x^3} + \frac{\log x}{x^3} = \frac{1}{x^3}.$$

$$\therefore A_1 = \frac{\frac{1}{x^2(1-x)^2} \times \frac{1}{x}}{\frac{1}{x^3}} = \frac{1}{(1-x)^2}$$

Integrating $A = \int \frac{dx}{(1-x)^2} + a = \int \frac{-d(1-x)}{(1-x)^2} + a = \frac{1}{(1-x)} + a$

$$B_1 = \frac{-\frac{1}{x^2(1-x)^2} \times \frac{\log x}{x}}{\frac{1}{x^3}} = -\frac{\log x}{(1-x)^2}$$

$$\therefore B = -\int \frac{\log x}{(1-x)^2} dx + b = -\log x \int \frac{dx}{(1-x)^2} + \int \left\{\frac{1}{x} \times \int \frac{dx}{(1-x)^2}\right\} dx + b$$

$$= -\log x \times \frac{1}{1-x} + \int \frac{1}{x} \cdot \frac{1}{1-x} dx + b$$

$$= -\frac{1}{1-x}\log x + \int \left(\frac{1}{x} + \frac{1}{1-x}\right) dx + b$$

or, $B = -\frac{1}{1-x}\log x + \log x - \log(1-x) + b$

$$\therefore y = Au + Bv$$

$$= \left(\frac{1}{1-x} + a\right)\frac{\log x}{x} + \left[-\frac{1}{1-x}\log x + \log x - \log(1-x) + b\right]\frac{1}{x}$$

$$= \frac{1}{x(1-x)}\log x + a\frac{\log x}{x} - \frac{1}{x(1-x)}\log x + \frac{\log x}{x} - \frac{\log(1-x)}{x} + \frac{b}{x}$$

or, $y = \frac{1}{x}[(a+1)\log x + b - \log(1-x)]$

§ 1.10 Riccati's Equation

The first order non-linear ordinary differential equation

$$\frac{dy}{dx} + P(x)y + Q(x)y^2 = R(x), \quad Q(x) \neq 0, \text{ -------(1)}$$

is known as the generalized Riccati's equation .

This can be reduced to a linear equation of the second order by the substitution

$$y = \frac{1}{Qu}\frac{du}{dx} \quad --- (2)$$

through which the dependent variable y is transformed to u.

Using (2), we obtain $\quad \frac{dy}{dx} = \frac{1}{Qu}\frac{d^2u}{dx^2} - \frac{1}{Qu^2}\left(\frac{du}{dx}\right)^2 - \frac{1}{Q^2u}\frac{dQ}{dx}\frac{du}{dx}$

Then equation (1) becomes

$$\frac{1}{Qu}\frac{d^2u}{dx^2} - \frac{1}{Qu^2}\left(\frac{du}{dx}\right)^2 - \frac{1}{Q^2u}\frac{dQ}{dx}\frac{du}{dx} + P(x)\frac{1}{Qu}\frac{du}{dx} + Q(x)\times\frac{1}{Q^2u^2}\left(\frac{du}{dx}\right)^2 = R(x)$$

or, $$\frac{d^2u}{dx^2} + \left(P - \frac{1}{Q}\frac{dQ}{dx}\right)\frac{du}{dx} - RQu = 0 \quad --- (3)$$

This is a linear equation of second order in u and can be attempted for a solution by the methods discussed earlier.

§ 1.11 Solved examples

Example 1. Solve $\frac{dy}{dx} + \frac{2}{x}y + \frac{1}{2}x^3y^2 = \frac{1}{2x}$ --- (1)

Solution: Here $P = \frac{2}{x}, Q = \frac{1}{2}x^3, R = \frac{1}{2x}$

Put $y = \frac{1}{Qu}\frac{du}{dx} = \frac{2}{x^3}\frac{1}{u}\frac{du}{dx}$

So that $\frac{dy}{dx} = \frac{2}{ux^3}\frac{d^2u}{dx^2} - \frac{2.3}{x^4}\frac{1}{u}\frac{du}{dx} - \frac{2}{x^3.u^2}\left(\frac{du}{dx}\right)^2$

Then the given equation (1) becomes

$$\frac{2}{ux^3}\frac{d^2u}{dx^2} - \frac{6}{x^4u}\frac{du}{dx} - \frac{2}{x^3.u^2}\left(\frac{du}{dx}\right)^2 + \frac{2}{x}\frac{2}{x^3}\frac{1}{u}\frac{du}{dx} + \frac{1}{2}x^3 \times \frac{4}{x^6u^2}\left(\frac{du}{dx}\right)^2 = \frac{1}{2x}$$

or, $$\frac{d^2u}{dx^2} - \frac{3}{x}\frac{du}{dx} - \frac{1}{u}\left(\frac{du}{dx}\right)^2 + \frac{2}{x}\frac{du}{dx} + \frac{1}{u}\left(\frac{du}{dx}\right)^2 = \frac{1}{2x} \times \frac{x^3u}{2}$$

or, $$\frac{d^2u}{dx^2} - \frac{1}{x}\frac{du}{dx} - \frac{x^2}{4}u = 0 \quad --- (2)$$

Here $P'(x) = -\frac{1}{x}$, $Q'(x) = -\frac{x^2}{4}$, $R' = 0$

To solve the equation (2) ,we change the independent variable x to z by the substitution

$$\frac{dz}{dx} = \sqrt{\frac{-Q'}{a^2}} = \sqrt{\frac{\frac{x^2}{4}}{\frac{1}{4}}} = \sqrt{x^2} = x, \text{ where } a^2 = \frac{1}{4}, \quad \text{so that}, \ z = \frac{x^2}{2}, \frac{d^2z}{dx^2} = 1$$

Then

$$P_1(x) = \frac{\frac{d^2z}{dx^2} + P'\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2} = \frac{1 + \left(-\frac{1}{x}\right)x}{x^2} = \frac{0}{x^2} = 0$$

$$Q_1(x) = \frac{Q'}{\left(\frac{dz}{dx}\right)^2} = \frac{-\frac{x^2}{4}}{x^2} = -\frac{1}{4}, R_1(x) = \frac{R'(x)}{\left(\frac{dz}{dx}\right)^2} = 0$$

Then the equation (2)reduces to $\frac{d^2u}{dz^2} - \frac{1}{4}u = 0$ or, $\left(D^2 - \frac{1}{4}\right)u = 0$

The auxiliary equation is $D^2 - \frac{1}{4} = 0 \Rightarrow D = \pm\frac{1}{2}$ $\quad \therefore u = C_1e^{\frac{1}{2}z} + C_2e^{-\frac{1}{2}z}$

Finally,

$$y = \frac{1}{Qu}\frac{du}{dx} = \frac{2}{x^3}\frac{\frac{1}{2}\left(C_1e^{\frac{1}{2}z} - C_2e^{-\frac{1}{2}z}\right)}{\left(C_1e^{\frac{1}{2}z} + C_2e^{-\frac{1}{2}z}\right)}\frac{dz}{dx} = \frac{2}{x^3}\times\frac{1}{2}\times x\times\frac{e^{\frac{1}{2}z} - \frac{C_2}{C_1}e^{-\frac{1}{2}z}}{e^{\frac{1}{2}z} + \frac{C_2}{C_1}e^{-\frac{1}{2}z}}$$

Finally, on putting, $\frac{C_2}{C_1} = k$ and replacing z by $\frac{x^2}{2}$ we get

$$y = \frac{1}{x^2}\frac{e^{\frac{x^2}{4}} - ke^{-\frac{x^2}{4}}}{e^{\frac{x^2}{4}} + ke^{-\frac{x^2}{4}}}$$

Example 2 Solve $\frac{dy}{dx} - (\tan x + 3\cos x)y + (\cos^2 x)y^2 = -2$ $\quad$ --- (1)

Solution: Here P=-(tanx+3cosx), Q=cos²x, R=-2

Putting $y' = \frac{1}{Qu}\frac{du}{dx} = \frac{\sec^2 x}{u}\frac{du}{dx}$

We get $\frac{du}{dx} = 2\sec x.\sec x\tan x \times \frac{1}{u}\frac{du}{dx} - \frac{\sec^2 x}{u^2}\left(\frac{du}{dx}\right)^2 + \frac{\sec^2 x}{u}\frac{d^2u}{dx^2}$

Then equation (1) becomes

$$\frac{\sec^2 x}{u}\frac{d^2u}{dx^2} + \frac{2\sec^2 x\tan x}{u}\frac{du}{dx} - \frac{\sec^2 x}{u^2}\left(\frac{du}{dx}\right)^2 -$$

$$(\tan x + 3\cos x)\times\frac{\sec^2 x}{u}\frac{du}{dx} + \cos^2 x \times \frac{\sec^4 x}{u^2}\left(\frac{du}{dx}\right)^2 = -2$$

or, $\frac{d^2u}{dx^2} + 2\tan x\frac{du}{dx} - \tan x\frac{du}{dx} - 3\cos x\frac{du}{dx} = -2u\cos^2 x$

or, $\frac{d^2u}{dx^2} + (\tan x - 3\cos x)\frac{du}{dx} + 2(\cos^2 x)u = 0 ------> (2)$

Here $P' = \tan x - 3\cos x$, $Q' = 2\cos^2 x$, $R' = 0$. To solve equation (2), we change the independent variable x to z by $\frac{dz}{dx} = \sqrt{\frac{Q'}{a^2}} = \sqrt{\frac{2\cos^2 x}{2}} = \cos x$ $\quad \therefore z = \sin x, \frac{d^2z}{dx^2} = -\sin x$ and

$$P_1(x) = \frac{\frac{d^2z}{dx^2} + P'\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2} = \frac{-\sin x + (\tan x - 3\cos x)\cos x}{\cos^2 x}$$

$$= \frac{-\sin x + \sin x - 3\cos^2 x)}{\cos^2 x} = \frac{3\cos^2 x}{\cos^2 x} = -3$$

$$Q_1(x) = \frac{Q'}{\left(\frac{dz}{dx}\right)^2} = \frac{2\cos^2 x}{\cos^2 x} = 2, \qquad R_1(x) = \frac{R'}{\left(\frac{dz}{dx}\right)^2} = 0$$

Then the equation (2) becomes

$$\frac{d^2u}{dz^2} - 3\frac{du}{dz} + 2u = 0 \qquad (3)$$

or $(D^2-3D+2)u=0$, where $D = \frac{d}{dz}$

The auxiliary equation is $D^2-3D+2=0$ Or, $(D-2)(D-1)=0$ Or, D=2 and 1

$\therefore u = C_1e^{2z} + C_2e^{z}$

Finally, we have the required solution of the given equation (1) as

$$y = \frac{1}{Qu}\frac{du}{dx} = \frac{1}{\cos^2 x}\,\frac{2C_1e^{2z} + C_2e^{z}}{C_1e^{2z} + C_2e^{z}}\,\frac{dz}{dx}$$

$$= \frac{2e^{2\sin x} + \dfrac{C_2}{C_1}e^{\sin x}}{e^{2\sin x} + \dfrac{C_2}{C_1}e^{\sin x}} \times \sec^2 x \times \cos x, \quad \text{put } \frac{C_2}{C_1} = k$$

$$\therefore\ y = \frac{2e^{2\sin x} + ke^{\sin x}}{e^{2\sin x} + ke^{\sin x}} \times \sec x$$

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UNIT 2

Solution in Series

2.1 Introduction: Consider the second order linear differential equation

$$a_0(x)\frac{d^2y}{dx^2}+a_1(x)\frac{dy}{dx}+a_2(x)y=0 \qquad (1)$$

Sometimes an equation of the form (1) may not be of any standard class which were discussed earlier in chapter 1. In that case, we may fail to obtain its solution in terms of known elementary functions which are made up of integral or fractional powers of x, sines and cosines, exponentials and logarithms such as

$$(1+2x)e^x,\ \sin x + x\cos x,\ x^{1/2}+x^{-1/2},\ x+\log x,\ e^{\frac{1}{x}} \text{ etc.}$$

The first and the second of these functions can be expanded by Maclaurin's theorem in ascending integral power of x. The others cannot be expanded in this form. Of course the last function can be expanded in powers of 1/x.

In such situations, (when the solution of equation (1) can not be obtained in terms of elementary functions), we must seek other methods of expression for the solution. One way of seeking the solution of equation (1) is to assume that this equation has a solution which can be expressed in the form of an infinite series, say

$$c_0+c_1(x-x_0)+c_2(x-x_0)^2+\cdots=\sum_{n=0}^{\infty}c_n(x-x_0)^n \qquad (2)$$

Where $c_0, c_1, c_2, \cdots c_n, \ldots$ are constants. Under this assumption, the coefficients $c_0, c_1, c_2, \cdots c_n, \ldots$ can be determined so that that the series (2) satisfies the equation (1)

The simplified form of equation (1) is

$$\frac{d^2y}{dx^2}+P_1(x)\frac{dy}{dx}+P_2(x)y=0 \qquad (3)$$

Where $p_1(x)=\dfrac{a_1(x)}{a_0(x)}$ and $p_2(x)=\dfrac{a_2(x)}{a_0(x)}$

The solution so obtained in terms of infinite series in powers of $(x-x_0)$ with the independent variable x, is known as power series solution of the given differential equation. Many important equations like those of Legendre, Bessel, and Gauss (or Hypergeometric) are solved by the method of series solution. An expression of the form (2) is called a power series.

Definition: A function f(x) is said to analytic at $x = x_0$ if its Taylor series expansion about x_0, namely

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n,$$

exists and converges to f(x) for all x in some open interval including x_0.

All polynomial functions are analytic everywhere .The function e^x, sin x and cos x are also analytic everywhere. A rational function like $p_1(x)$ and $p_2(x)$ given above is analytic everywhere except at those values of x at which the denominator $a_0(x)$ of the rational function is zero.

Ordinary point and singular point

The point x_0 is called an **ordinary point** of the differential equation (1) if both the functions $p_1(x)$ and $p_2(x)$ in the equivalent normalized form (3) are analytic at x_0. If either (or both) of these function is not analytic at x_0, then x_0 is called a **singular point** of the differential equation (1).

Illustration: Consider the differential equation

$$(x-1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} + \frac{1}{x}y = 0 \qquad \rightarrow (4)$$

Its normalized form is $\dfrac{d^2y}{dx^2} + \dfrac{x}{x(x-1)}\dfrac{dy}{dx} + \dfrac{1}{x(x-1)}y = 0$

Here $p_1(x) = \dfrac{x}{x-1}$, $p_2(x) = \dfrac{1}{x(x-1)}$

The function $p_1(x)$ is analytic everywhere except at x = 1 and the function $p_2(x)$ is analytic everywhere except at x = 0 and 1. Thus x = 0 and x = 1 are the singular points of the differential equation (4). All other points are ordinary points.

Theorem: (for existence of power series solution)

If the point x_0 is an ordinary point of the linear differential equation (1), then this equation (of second order) has two non-trivial linearly independent power series solutions of the form, $\sum_{n=0}^{\infty} c_n(x - x_0)^n$ and these power series converge in some interval $|x - x_0| < R$ (where $R > 0$) about x_0.

The general solution of the equation (1) about the ordinary point x_0 may be obtained as a linear combination of these two linearly independent solutions.

2.2 Power series solution about an ordinary point.

Here we give the procedure for obtaining the power series solution of a linear differential equation about one of its ordinary point through the following examples

Ex. 1 Find the power series solution of the differential equation

$$\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2+2)y = 0 \qquad (1) \text{ in powers of x about the point } x = 0.$$

Solution : Here $p_1(x) = x$, $p_2(x) = x^2+2$, which are analytic functions of x for all values of x including the point x = 0.

So, two linearly independent power series solutions of the equation (1) about x = 0 should exist.

Let a solution be $$y = \sum_{n=0}^{\infty} c_n(x-0)^n = \sum_{n=0}^{\infty} c_n x^n \qquad (2)$$

so that $$\frac{dy}{dx} = \frac{d}{dx}\{c_0 + c_1x + c_2x^2 + \cdots c_nx^n + \cdots\}$$

$$= c_1 + 2c_2x + 3c_3x^2 + \cdots + n\,c_nx^{n-1} + \cdots = \sum_{n=1}^{\infty} c_n.nx^{n-1}$$

$$\frac{d^2y}{dx^2} = c_2.2.1 + c_3.3.2x + c_4.4.3x^2 + \cdots + c_n.n.(n-1)x^{n-2} + \cdots = \sum_{n=2}^{\infty} c_n.n(n-1)x^{n-2}$$

putting these expressions in equation (1) we get

$$\sum_{n=2}^{\infty} c_n.n(n-1)x^{n-2} + x\sum_{n=1}^{\infty} c_n.nx^{n-1} + (x^2+2)\sum_{n=0}^{\alpha} c_nx^n = 0$$

$$\text{or,} \quad \sum_{n=2}^{\infty} c_n.n(n-1)x^{n-2} + \sum_{n=1}^{\infty} c_n.nx^n + \sum_{n=0}^{\alpha} c_nx^{n+2} + 2\sum_{n=0}^{\alpha} c_nx^n = 0 \cdots (3)$$

Now, $\sum_{n=2}^{\infty} n(n-1)\, c_nx^{n-2}$, put $m = n - 2$ $\therefore$ when $n = 2, m = 0$

$$= \sum_{m=0}^{\infty} (m+2)(m+1)\, c_{m+2}x^m$$

$$= \sum_{n=0}^{\infty} (n+2)(n+1)\, c_{n+2}x^n \quad \because \text{m is a dummy suffix } \therefore m \to n.$$

The 3rd term = $\sum_{n=0}^{\infty} c_nx^{n+2}$, put $m = n + 2$

$$= \sum_{m=2}^{\infty} c_{m-2}x^m = \sum_{n=2}^{\infty} c_{n-2}x^n,$$

Then the equation (3) reduces to

$$\sum_{n=0}^{\infty} (n+2)(n+1)\, c_{n+2}x^n + \sum_{n=1}^{\alpha} n\, c_n x^n + \sum_{n=2}^{\alpha} c_{n-2}x^n + 2\sum_{n=0}^{\alpha} c_n x^n = 0$$

$$\text{or } 2.1.c_2x^0 + 3.2.c_3x^1 + \sum_{n=2}^{\infty} (n+2)(n+1)\, c_{n+2}x^n + 1.c.x^1 + \sum_{n=2}^{\alpha} n\, c_n x^n$$

$$+ \sum_{n=2}^{\infty} c_{n-2}x^n + 2.c_0.x^0 + 2.c_1.x^1 + 2\sum_{n=2}^{\alpha} c_n x^n = 0$$

$$\text{or, } (2.c_2 + 2.c_0) + (6.c_3 + c_1 + 2.c_1)\,x + \sum_{n=2}^{\alpha}\left[(n+2)(n+1)\, c_{n+2} + n\, c_n + c_{n-2} + 2\, c_n\right]x^n = 0$$

$$\text{or } (2c_0 + 2c_2) + (3c_1 + 6c_3)x + \sum_{n=2}^{\infty}\left[(n+2)(n+1)c_{n+2} + (n+2)c_n + c_{n-2}\right]x^n = 0 \quad \text{....(4)}$$

Since the relation (4) is valid for all x in the interval of convergence $|x-0| < R$, the coefficient of each power of x in the left hand member of equation (4) must vanish. This leads to

$$2c_0 + 2c_2 = 0 \Rightarrow c_2 = -c_0 \qquad \text{....(5)}$$

$$3c_1 + 6c_3 = 0 \Rightarrow c_3 = -\frac{1}{2}c_1 \qquad \text{....(6)}$$

$$\text{and} \quad (n+2)(n+1)c_{n+2} + (n+2)c_n + c_{n-2} = 0,\ n \ge 2 \qquad \text{....(7)}$$

The condition (7) is called the **recurrence formula**. Through this relation, we can express each coefficient c_{n+2} for $n \ge 2$ in terms of all the previous coefficients c_n and c_{n-2}.

$$\text{Thus from (7)} \qquad c_{n+2} = -\frac{(n-2)c_n + c_{n-2}}{(n+1)(n+2)},\ n \ge 2 \qquad \text{....(8)}$$

for n = 2, (8) gives

$$c_4 = -\frac{4c_2 + c_0}{3.4} = -\frac{4(-c_0) + c_0}{3.4}, \quad \text{using (5)}$$

$$\text{or} \quad c_4 = \frac{1}{4}c_0 \qquad \text{......(9)}$$

$$\text{for } n = 3, \quad c_5 = -\frac{5c_3 + c_1}{4.5} = -\frac{5\left(-\frac{1}{2}c_1\right) + c_1}{4.5}, \quad \text{using (6)}$$

$$\text{or } c_5 = +\frac{3}{40}c_1 \quad \text{....(10)}$$

Putting these expressions for the coefficients in the assumed solution (2), we get

$$y = c_0 + c_1x - c_0x^2 - \frac{1}{2}c_1x^3 + \frac{1}{4}c_0x^4 + \frac{3}{40}c_1x^5 + \cdots$$

$$= c_0\left(1 - x^2 + \frac{1}{4}x^4 + \cdots\right) + c_1\left(x - \frac{1}{2}x^3 + \frac{3}{40}x^5 + \cdots\right) \quad \text{....(11)}$$

This gives the solution of the given equation (1) in powers of x near the ordinary point x=0. The two series in bracket in (11) are the power series expansions of two linearly independent solutions of equation (1), and c_0 and c_1 are arbitrary constants. Thus the combination of the series in (11) represent the general solution of the differential equation (1) in powers of x.

Exercise 2: The differential equation

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0 \quad \text{....(1)}$$

(where n is a constant) is called **Legendre's differential equation** of order n.

(a) Show that x = 0 is an ordinary point of this differential equation and find two independent power series solution in powers of x.

(b) Show that if n is a non-negative integer, then one of the two solutions found in part (a) is a polynomial of degree n.

Solution: Part (a) The normalised form of equation (1) is

$$\frac{d^2y}{dx^2} - \frac{2x}{(1-x^2)}\frac{dy}{dx} + \frac{n(n+1)}{(1-x^2)}y = 0 \quad \text{....(2)}$$

Here $P_1(x)]_{x=0} = \left.\frac{-2x}{1-x^2}\right]_{x=0} = 0$ and $P_2(x)]_{x=0} = \left.\frac{n(n+1)}{1-x^2}\right]_{x=0} = n(n+1)$

Thus x = 0 is an arbitrary point of the given differential equation .

Let us assume its solution as

$$y = \sum_{r=0}^{\infty} a_r x^r = a_0 + a_1x + a_2x^2 + \cdots + a_rx^r + \cdots$$

$$\therefore \frac{dy}{dx} = a_1 + 2a_2x + 3a_3x^2 + \cdots + ra_rx^{r-1} + \cdots = \sum_{r=1}^{\infty} ra_rx^{r-1}$$

$$\frac{d^2y}{dx^2} = 2.1.a_2 + 3.2a_3x + 4.3a_4x^2 + \cdots + (r)(r-1)a_rx^{r-2} + \cdots = \sum_{r=2}^{\infty} r(r-1)a_rx^{r-2}$$

putting these expressions in equation (1) we get

$$\left(1-x^2\right)\sum_{r=2}^{\infty} r(r-1)a_r x^{r-2} - 2x\sum_{r=1}^{\infty} ra_r x^{r-1} + n(n+1)\sum_{r=0}^{\infty} a_r x^r = 0$$

$$\text{or, } \sum_{r=2}^{\infty} r(r-1)a_r x^{r-2} - \sum_{r=1}^{\infty} r(r-1)a_r x^r - 2\sum_{r=1}^{\infty} ra_r x^r + n(n+1)\sum_{r=0}^{\infty} a_r x^r = 0 \quad \ldots(3)$$

Here first term $= \sum_{r=2}^{\infty} r(r-1)a_r x^{r-2}$, putting m = r-2 so that m = 0 when r = 2

$$= \sum_{m=0}^{\infty}(m+2)(m+1)a_{m+2}x^m$$

$$= \sum_{r=0}^{\infty}(r+2)(r+1)a_{r+2}x^r \qquad \rightarrow \because \text{ r is dummy suffix.}$$

then equation (3) becomes

$$\sum_{r=0}^{\infty}(r+2)(r+1)a_{r+2}x^r - \sum_{r=2}^{\infty} r(r-1)a_r x^r - 2\sum_{r=1}^{\infty} ra_r x^r + n(n+1)\sum_{r=0}^{\infty} a_r x^r = 0$$

$$\text{or} \quad 2.1a_2 + 3.2a_3 x + \sum_{r=2}^{\infty}(r+2)(r+1)a_{r+2}x^r - \sum_{r=2}^{\infty} r(r-1)a_r x^r - 2.1.a_1 x^1 -$$

$$2\sum_{r=2}^{\infty} ra_2 x^r + n(n+1)a_0 + n(n+1)a_1 x + n(n+1)\sum_{r=2}^{\infty} a_r x^r = 0$$

$$\text{or} \quad \{2a_2 + n(n+1)a_0\} + \{6a_3 - 2a_1 + n(n+1)a_1\}x + \sum_{r=2}^{\infty}[(r+2)(r+1)a_{r+2} - r(r-1)a_r - 2ra_r + n(n+1)a_r]x^r = 0 \quad \ldots(4)$$

Equating to zero the coefficients of various powers of x on the left hand side of equation (4) we obtain the following relations :

$$x^0 : \rightarrow 2a_2 + n(n+1)a_0 = 0 \Rightarrow a_2 = -\frac{1}{2}n(n+1)a_0$$

$$x^1 : \rightarrow 6a_3 - 2a_1 + n(n+1)a_1 = 0 \Rightarrow a_3 = \frac{1}{6}[2-n(n+1)]a_1$$

$$= -\frac{a_1}{3!}(n-1)(n+2)$$

The recurrence formula for $n \geq 2$ is

$$(r+2)(r+1)a_{r+2} - r(r-1)a_r - 2ra_r + n(n+1)a_r = 0$$

or $\quad a_{r+2} = \dfrac{r(r-1)+2r-n(n+1)}{(r+2)(r+1)} a_r$

or $\quad a_{r+2} = \dfrac{(r+n+1)(r-n)}{(r+2)(r+1)} a_r \qquad \ldots(5)$

Putting r = 2 and 3 in (5) and simplifying we get

$$a_4 = \frac{(n-2).n.(n+1)(n+3)}{4!} a_0 \qquad \text{and} \qquad a_5 = \frac{(n-3)(n-1)(n+2)(n+4)}{5!} a_1$$

Putting these expressions for the coefficients in the assumed solution we obtain

$$y = a_0\left[1 - \frac{n(n+1)}{2!}x^2 + \frac{(n-2)n(n+1)(n+3)}{4!}x^4 - \cdots\right] + a_1\left[x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-3)\ (n-1)\ (n+2)\ (n+4)}{5\,!}x^5 - \cdots\right] \quad (6)$$

Each power series within the brackets in the relation (6) is a solution of the given Legendre's equation and they are linearly independent of each other. Thus (6) gives the complete solution of the differential equation (1).

Part (b): When n is a non-negative integer, then because of the factor (r-n) in the recurrence formula (5) a set of coefficients will be zero beyond the term corresponding to r = n. When n is odd, the second series in (6) will be terminated beyond the term corresponding to x^r. Thus the second series will be reduced to a polynomial of degree n only.

Similarly, when n is even, the first series in (6) will be a polynomial of degree n.

Exercise 3: Solve $y'' + (x-1)y' + y = 0$ in powers of x - 2.

Hints: Put v = x - 2 to convert the given differential equation to

$$\frac{d^2y}{dv^2} + (v+1)\frac{dy}{dv} + y = 0$$

The series solution of this equation about v = 0 may be obtained and then replacing v by (x-2), the required solution is found as

$$y = a_0\left[1 - \frac{1}{2}(x-2)^2 + \frac{1}{6}(x-2)^3 + \frac{1}{12}(x-2)^4 - \frac{1}{20}(x-2)^5 - \frac{1}{180}(x-2)^6 + \cdots\right]$$
$$+ a_1\left[(x-2) - \frac{1}{2}(x-2)^2 - \frac{1}{6}(x-2)^3 + \frac{1}{6}(x-2)^4 - \frac{1}{36}(x-2)^6 + \cdots\right]$$

2.3 Solution About Singular Points; The Method Of Frobenius:

Suppose x_0 is a singular point of the linear differential equation

$$a_0(x)\frac{d^2y}{dx^2}+a_1(x)\frac{dy}{dx}+a_2(x)y=0 \quad .....(1)$$

In this case we are not assured of a power series solution of equation (1) in the form

$$y=\sum_{n=0}^{\infty}c_n(x-x_0)^n \quad ....(2)$$

Such a solution about a singular point, in general, does not exists.

Illustration: For the equation

$$x^2y''+(x^2-x)y'+2y=0 \quad ...(3)$$

$x=0$ is a singular point since $a_0(x)\big]_{x=0}=x^2\big]_{x=0}=0$.

If we assume a solution of the from

$$y=c_0+c_1x+c_2x^2+c_3x^3+c_4x^4+.... \quad ....(4)$$

and substitute this in the given equation (3), we get

$$x^2(2c_2+3.2c_3x+4.3c_4x^2+5.4\,c_5x^3+...)+(x^2-x)(c_1+2c_2x+3c_3x^2+4c_4x^3+...)$$
$$+2(c_0+c_1x+c_2x^2+c_3x^3+c_4x^4+...)=0$$

On simplification this reduces to

$$2c_0+c_1x+(2c_2+c_1)x^2+(5c_3+2c_2)x^3+(10c_4+3c_3)x^4+...=0$$

If the solution is to be satisfied identically, it requires that

$$c_0=0,\ c_1=0,\ c_2=-\frac{1}{2}c_1=0,\ c_3=-\frac{2}{5}c_2=0 \quad \text{etc.}$$

Therefore, there is no series of the form (4) satisfying the equation (3). So it requires to obtain a different type of series as a trial solution of the given differential equation about a singular point $x = x_0$.

Definition: The singular point $x = x_0$ of the differential equation (1) is called a **regular singular point**, if the functions $(x-x_0)P_1(x)$ and $(x-x_0)^2P_2(x)$ .....(5)

where $P_1(x)=\dfrac{a_1(x)}{a_0(x)}$ and $P_2(x)=\dfrac{a_2(x)}{a_0(x)}$

are both analytic at x_0 i.e. the functions in (5) can be expanded in Taylor series about $x = x_0$.

If either (or both) of the functions defined in (5) is not analytic at $x = x_0$, then x_0 is called an **irregular singular point** of the differential equation (1).

Illustration 1: Consider the differential equation

$$(1+x)y'' + 2xy' - 3y = 0$$

Here $a_0(x) = 1 + x$, $a_1(x) = 2x$, $a_2(x) = -3$

At $1 + x = 0$ i.e. $x = -1$, $a_0(x)$ and so $x = -1$ is a singular point of the given differential equation .

Also $P_1(x) = \dfrac{a_1(x)}{a_0(x)} = \dfrac{2x}{1+x}$, $P_2(x) = \dfrac{a_2(x)}{a_0(x)} = -\dfrac{3}{1+x}$

Now $(x-(-1))P_1(x) = (x+1)\dfrac{2x}{1+x} = 2x$ which is analytic at $x = -1$

and $(x-(-1))^2 P_2(x) = (x+1)^2 \dfrac{(-3)}{(1+x)} = -3(1+x)$ which is analytic at $x = -1$

$\therefore x = -1$ is a regular singular point of the given differential equation .

Illustration 2: Consider the differential equation

$$x^2(x-2)^2\frac{d^2y}{dx^2} + 2(x-2)\frac{dy}{dx} + (x+1)y = 0 \quad \text{or} \quad \frac{d^2y}{dx^2} + \frac{2}{x^2(x-2)}\frac{dy}{dx} + \frac{x+1}{x^2(x-2)^2}y = 0$$

Here $P_1(x) = \dfrac{2}{x^2(x-2)}$ and $P_2(x) = \dfrac{x+1}{x^2(x-2)^2}$

The singular points of the differential equation are $x = 0$ and $x = 2$.

Let us first consider the singular point $x = 0$.

The function $R_1(x) = xP_1(x) = \dfrac{2}{x(x-2)}$ is not-analytic at $x = 0$

where as $R_2(x) = x^2P_2(x) = \dfrac{x+1}{(x-2)^2}$ is analytic at $x = 0$.

So $x = 0$ is an **irregular singular point** of the given equation.

Next, consider the singular point $x = 2$.

In this case $R_1(x) = (x-2)P_1(x) = \dfrac{2}{x^2}$ is analytic at $x = 2$

and $R_2(x) = (x-2)^2P_2(x) = \dfrac{x+1}{x^2}$ is also analytic at $x = 2$.

Hence $x = 2$ is a **regular singular point** of the given differential equation.

Theorem: If x_0 is a regular singular point of the differential equation

$$a_0(x)\frac{d^2y}{dx^2}+a_1(x)\frac{dy}{dx}+a_2(x)y=0 \qquad ....(1)$$

then this differential equation has **at least one** non trivial solution of the form

$$y=(x-x_0)^{\rho}\sum_{r=0}^{\infty}c_r(x-x_0)^r \qquad .....(2)$$

where ρ is a definite (real or complex) constant which may be determined, and this solution is valid in some interval $0<|x-x_0|<R$ (where $R>0$) about x_0.

2.4 The method of Frobenius:

The procedure of finding a solution of the differential equation of the type (1) given above, about a regular singular point x_0 starting from the assumption of a series solution of the type (2) is known as the **method of Frobenius**. The procedure is laid down as under.

Procedure: Let the solution be $y=(x-x_0)^{\rho}\sum_{r=0}^{\infty}c_r(x-x_0)^r=\sum_{r=0}^{\infty}c_r(x-x_0)^{\rho+r}$ (3)

where $c_0\neq 0$ and x_0 is a regular singular point of the differential equation (1).

Assuming term by term differentiation of the series (3) we get

$$\frac{dy}{dx}=\sum_{r=0}^{\infty}(\rho+r)c_r(x-x_0)^{\rho+r-1} \qquad (4)$$

$$\frac{d^2y}{dx^2}=\sum_{r=0}^{\infty}(\rho+r)(\rho+r-1)c_r(x-x_0)^{\rho+r-2} \qquad (5)$$

On substituting these expressions for y, y', and y'', the differential equation (1) becomes

$$a_0(x)\sum_{r=0}^{\infty}(\rho+r)\ (\rho+r-1)c_r(x-x_0)^{\rho+r-2}+a_1(x)\sum_{r=0}^{\infty}(\rho+r)\ \rho_r(x-x_0)^{\rho+r-1}$$

$$+a_2(x)\sum_{r=0}^{\infty}c_r(x-x_0)^{\rho+r}=0$$

On rearranging the terms on the left hand side in ascending powers of $(x - x_0)$, this equation finally takes the form

$$K_0\,(x-x_0)^{\rho+k}+K_1(x-x_0)^{\rho+k+1}+K_2(x-x_0)^{\rho+k+2}+\cdots=0 \qquad (6)$$

where k is a certain integer and the coefficients K_i, $i = 0, 1, 2, \ldots$ are functions of ρ and certain of the coefficients c_r, of the solution (3).

In order that (6) is valid for all x in $0 < |x - x_0| < R$ we must set

$$K_0 = K_1 = K_2 = \ldots = 0.$$

The coefficient K_0 of lowest power $\rho + k$ of $(x - x_0)$ equated to zero gives $K_0 = 0$ ….(7).

This is a quadratic equation in ρ, called the **indicial equation** of the differential equation (1). The two roots ρ_1, ρ_2 $(\rho_1 \geq \rho_2)$ of this indicial equation are called exponents of the equation (1) and are the only possible values of ρ in the assumed solution (3).

The other equations $K_1 = 0, K_2 = 0 \cdots$ (8)

allow us to determine the various coefficients c_r in terms of ρ and other preceding coefficients.

We now substitute the larger root ρ_1 for ρ and determine c_r for $r = 1,2,\ldots,n,\ldots$ to satisfy the conditions (8). When c_r are so chosen, the resulting series (3) is a solution of the desired form.

If $\rho_2 \neq \rho_1$, we may repeat the earlier procedure by using the smaller root ρ_2 instead of ρ_1. In this way a second series solution of the desired form (3) is obtained. However, this second series solution may not be linearly independent of the earlier solution with $\rho = \rho_1$. In that case, we have to search for a second solution linearly independent of the first one.

It will be seen in discussing the following examples that when the roots of the indicial equation are equal or differ by a positive integer, then the second solution is to be obtained in a complicated form to make it linearly independent of the first solution.

2.5 Examples of various cases of Frobenius method.

Now we consider the various cases of the roots of the indicial equation while obtaining series solution of linear differential equations by the method of Frobenius, through some examples.

Case I) The roots of the indicial equation unequal and differing by a quantity which is not an integer.

Exercise 1. Use the method of Frobenius to find solutions near x = 0 of the differential equation

$$2x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + (x-5)y = 0 \qquad (1)$$

Solution: Obviously x = 0 is a singular point of the equation (1).

Here $\quad P_1(x) = -\dfrac{x}{2x^2} = -\dfrac{1}{2x}, \qquad P_2(x) = \dfrac{x-5}{2x^2}.$

$$R_1(x) = (x-0)\,P_1(x) = -\frac{x}{2x} = -\frac{1}{2} \text{ which is analytic at } x = 0.$$

$$R_2(x) = (x-0)^2\, P_2(x) = x^2\,\frac{(x-5)}{2x^2} = \frac{(x-5)}{2} \quad \text{which is analytic at } x = 0$$

Hence, $x = 0$ is a regular singular point of the differential equation (1).

So, we assume the series solution

$$\left.\begin{aligned} & y = \sum_{r=0}^{\infty} c_r x^{\rho+r} = x^{\rho}(c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots), \qquad c_0 \neq 0 \\ & \text{so that} \quad \frac{dy}{dx} = \sum_{r=0}^{\infty} (\rho+r)\, c_r x^{\rho+r-1} \\ & \text{and} \quad \frac{d^2y}{dx^2} = \sum_{r=0}^{\infty} (\rho+r)(\rho+r-1)\, c_r x^{\rho+r-2} \end{aligned}\right\} \quad (2)$$

On substituting these expressions, the given differential equation reduces to

$$2x^2 \sum_{r=0}^{\infty} (\rho+r)(\rho+r-1) c_r x^{\rho+r-2} - x\sum_{r=0}^{\infty} (\rho+r)\, c_r x^{\rho+r-1} + (x-5)\sum_{r=0}^{\infty} c_r x^{\rho+r} = 0$$

or $\quad \sum_{r=0}^{\infty} [2(\rho+r)(\rho+r-1) - (\rho+r) - 5] c_r x^{\rho+r} + \sum_{r=0}^{\infty} c_r x^{\rho+r+1} = 0$

Now the last term in the equation is

$$\sum_{r=0}^{\infty} c_r x^{\rho+r+1} = \sum_{m=1}^{\infty} c_{m-1} x^{\rho+m}, \text{ put } m = r+1 \text{ when } r = 0,\ m = 1$$

$$= \sum_{r=1}^{\infty} c_{r-1} x^{\rho+r} \qquad \because m \text{ is dummy suffix.}$$

So we have

$$\sum_{r=0}^{\infty} [2(\rho+r)(\rho+r-1) - (\rho+r) - 5] c_r x^{\rho+r} + \sum_{r=1}^{\infty} c_{r-1} x^{\rho+r} = 0$$

or, $[2\rho(\rho-1) - \rho - 5] c_0 x^{\rho} + \sum_{r=1}^{\infty} \{[2(\rho+r)(\rho+r-1) - (\rho+r) - 5] c_r + c_{r-1}\} x^{\rho+r} = 0 \quad \ldots(3)$

The lowest power of x in (3) is x^{ρ}.

Its coefficient equated to zero gives us the indicial equation

$$[2\rho(\rho-1) - \rho - 5] c_0 = 0$$

or, $\quad 2\rho^2 - 3\rho - 5 = 0 \qquad \because c_0 \neq 0.$

The roots are $\rho_1 = \frac{5}{2}$ and $\rho_2 = -1$

$$\rho = \frac{3 \pm \sqrt{9-4.2.(-5)}}{2.2} = \frac{5}{2}, -1$$

These are the exponents of the differential equation (1) and are the only possible values of ρ in the assumed series solution (2). Equating to zero the coefficient of $x^{\rho+r}$ in (4), we get the recurrence formula

$$[2(\rho+r)(\rho+r-1)-(\rho+r)-5]c_r + c_{r-1} = 0, \qquad r \geq 1 \qquad \cdots(4)$$

On putting $\rho = \rho_1 = 5/2$ in (4), we get

$$\left[2\left(r+\frac{5}{2}\right)\left(r+\frac{3}{2}\right)-\left(r+\frac{5}{2}\right)-5\right]c_r + c_{r-1} = 0$$

or, $r(2r+7)c_r + c_{r-1} = 0$

$$\left(r+\frac{5}{2}\right)(2r+3-1)-5$$
$$= (r+5/2)(2r+2)-5$$
$$= (2r+5)(r+1)-5$$
$$= 2r^2+7r+5-5$$
$$= r(2r+7)$$

or, $c_r = -\frac{c_{r-1}}{r(2r+7)}, \quad r \geq 1 \qquad \ldots.(5)$

From (5), we get, by putting r = 1, 2, 3, 4 etc.

$$c_1 = -\frac{c_0}{1(2+7)} = -\frac{c_0}{9}, \quad c_2 = -\frac{c_1}{2(4+7)} = -\frac{1}{22} \times -\frac{c_0}{9} = \frac{c_0}{198}$$

$$c_3 = -\frac{c_2}{3(6+7)} = -\frac{1}{39} \times \frac{c_0}{198} = -\frac{c_0}{7722} \qquad \text{etc.}$$

Thus putting $\rho = 5/2$ in (2) and using these values of the coefficients, we obtain the solution $y = c_0 x^{5/2}\left(1 - \frac{1}{9}x + \frac{1}{198}x^2 - \frac{1}{7722}x^3 + \cdots\right)$ ....(6) corresponding to the larger root $\rho = \rho_1 = 5/2$.

We now put $\rho = -1$ in the recurrence formula (4) to obtain

$$\{2(r-1)(r-2)-(r-1)-5\}c_r + c_{r-1} = 0, \qquad r \geq 1$$

or, $r(2r-7)c_r + c_{r-1} = 0$

or, $c_r = -\dfrac{c_{r-1}}{r(2r-7)}$, $r \geq 1$ ....(7)

$2r^2 - 6r + 4 - r + 1 - 5$

$= 2r^2 - 7r + 0$
$= r(2r-7)$

This gives for r = 1, 2, 3 etc.

$$c_1 = \frac{c_0}{1(2-7)} = \frac{c_0}{5}, \quad c_2 = -\frac{c_1}{2(4-7)} = \frac{1}{6}\cdot\frac{c_0}{5} = \frac{c_0}{30}$$

$$c_3 = -\frac{c_2}{3(6-7)} = \frac{1}{3}\times\frac{c_0}{30} = \frac{c_0}{90}, \quad c_4 = -\frac{c_3}{4(8-7)} = -\frac{1}{4}\cdot\frac{c_0}{90} = -\frac{c_0}{360} \text{ etc.}$$

Thus putting ρ = -1 in the series (2) and inserting these values of the coefficients, we obtain the second solution

$$y = c_0 x^{-1}\left(1 + \frac{1}{5}x + \frac{1}{30}x^2 + \frac{1}{90}x^3 - \frac{1}{360}x^4 + \cdots\right) \quad ....(8)$$

The two solutions (6) and (8) corresponding to the two exponents $\rho = 5/2$ and -1 respectively are linearly independent. So putting $c_0 = a$ in (6) and $c_0 = b$ in (8) we may put the general solution in the form

$$y = ax^{5/2}\left(1 - \frac{1}{9}x + \frac{1}{198}x^2 - \frac{1}{7722}x^3 + \cdots\right) + bx^{-1}\left(1 + \frac{1}{5}x + \frac{1}{30}x^2 + \frac{1}{90}x^3 - \frac{1}{360}x^4 + \cdots\right)$$

where a and b are arbitrary constants.

In general, if indicial equation has two unequal roots ∝ and β differing by a quantity which is not an integer, we get two independent solutions by substituting these values of ρ in the assumed series for y.

Ex. 2 : Find the solution in series for Bessel's equation of order n,

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0 \quad ...(1) \quad \text{taking 2n as non integral.}$$

Solution : It is easily seen that x = 0 is a regular singular point of the differential equation (1). So, we assume the series solution

$$y = x^\rho \sum_{r=0}^{\infty} c_r x^r = \sum_{r=0}^{\infty} c_r x^{\rho+r}, \quad c_0 \neq 0 \quad ....(4)$$

so that $\dfrac{dy}{dx} = \sum_{r=0}^{\infty}(\rho+r)c_r x^{\rho+r-1}$ and $\dfrac{d^2y}{dx^2} = \sum_{r=0}^{\infty}(\rho+r)(\rho+r-1)c_r x^{\rho+r-2}$

On putting these expressions, the equation (1) becomes

$$\sum_{r=0}^{\infty}(\rho+r)(\rho+r-1)c_r x^{\rho+r}+\sum_{r=0}^{\infty}(\rho+r)c_r x^{\rho+r}+\sum_{r=0}^{\infty}c_r x^{\rho+r+2}-n^2\sum_{r=0}^{\infty}c_r x^{\rho+r}=0$$

Hence $\sum_{r=0}^{\infty}c_r x^{\rho+r+2}=\sum_{m=2}^{\infty}c_{m-2}x^{\rho+m}=\sum_{r=2}^{\infty}c_{r-2}x^{\rho+r}$ | $m=r+2$, when $r=0,\ m=2$

$$\text{Or,}\quad \sum_{r=0}^{\infty}\left[(\rho+r)(\rho+r-1)+(\rho+r)-n^2\right]c_r x^{\rho+r}+\sum_{r=2}^{\infty}c_{r-2}x^{\rho+r}=0$$

$$\text{Or,}\quad \{\rho(\rho-1)+\rho-n^2\}c_0 x^{\rho}+\{(\rho+1)(\rho+1-1)+(\rho+1)-n^2\}c_1 x^{\rho+r}$$

$$+\sum_{r=2}^{\infty}\left[(\rho+r)(\rho+r-1)+(\rho+r)-n^2\right]c_r x^{\rho+r}+\sum_{r=2}^{\infty}c_{r-2}x^{\rho+r}=0$$

$$\text{Or,}\quad (\rho^2-n^2)c_0 x^{\rho}+\left[(\rho+1)^2-n^2\right]c_1 x^{\rho+1}+\sum_{r=2}^{\infty}\left[\{(\rho+r)^2-n^2\}c_r+c_{r-2}\right]x^{\rho+r}=0 \quad .....(3)$$

Equating to zero the coefficient of lowest power of x i.e. of x^{ρ} in (3) we get the indicial equation $(\rho^2-n^2)c_0=0$, where $c_0\neq 0$

$$\therefore \rho^2-n^2=0 \text{ or } \rho=\pm n \quad ....(4)$$

Thus $\rho_1=n$ and $\rho_2=-n$

Since $\rho_1-\rho_2=2n$ is given to be non-integral, we expect two series solution of the form (2) to the given differential equation and these solutions should be linearly independent of each other. Next putting the coefficient of $x^{\rho+1}$ to zero we get

$$\left[(\rho+1)^2-n^2\right]c_1=0 \qquad ....(5)$$

Since the only possible values of ρ are given by (4) we have $(\rho+1)^2-n^2\neq 0$

Then equation (5) requires that $c_1=0$ ....(6)

Next, equating to zero the coefficient of $x^{\rho+r}$, $r\geq 2$, we obtain the recurrence formula

$$\{(\rho+r)^2-n^2\}c_r+c_{r-2}=0 \quad \text{Or,}\quad c_r=-\frac{c_{r-2}}{(\rho+r+n)(\rho+r-n)},\ r\geq 2 \qquad ....(7)$$

From this recurrence formula and the result (6), it is clear that all the odd coefficients i.e. $c_3=c_5=c_7=...=c_{2r+1}=0$.

We now put $\rho=\rho_1=n$ in (7) and obtain the first series solutions corresponding to the larger root of the indicial equation.

Putting r = 2, 4, 6 etc. and $\rho = n$ in (7),

$$c_2 = -\frac{c_0}{(n+2+n)(n+2-n)} = -\frac{c_0}{2(n+1).2} = -\frac{c_0}{2^2(n+1)}$$

$$c_4 = -\frac{c_2}{(n+4+n)(n+4-n)} = -\frac{1}{2(n+2).2^2}\left\{-\frac{c_0}{2(n+1).2}\right\} = \frac{c_0}{2^4.2!(n+1)(n+2)}$$

$$c_6 = -\frac{c_4}{(n+6+n)(n+6-n)} = -\frac{1}{2(n+3).2.3}\times\frac{c_0}{2^4.2!(n+1)(n+2)}$$

$$= -\frac{c_0}{2^6.3!(n+1)(n+2)(n+3)}$$

Similarly, $c_8 = \frac{c_0}{2^8.4!(n+1)(n+2)(n+3)(n+4)}$ and $c_{2r} = (-1)^r\frac{c_0}{2^{2r}.r!(n+1)(n+2)\cdots(n+r)}$

Then the first series solution of equation (1) is obtained on putting $\rho = n$ and $c_0 = \alpha$ as

$$y = a\left[x^n - \frac{x^{2+n}}{2^2(n+1)} + \frac{x^{4+n}}{2^4.2!(n+1)(n+2)} - \frac{x^{6+n}}{2^6.3!(n+1)(n+2)(n+3)} + \cdots\right.$$

$$\left.\cdots + (-1)^r\frac{x^{n+2r}}{2^{2r}.r!(n+1)(n+2)\cdots(n+r)} + \cdots\right]$$

or $y_1(x) = a\sum_{r=0}^{\infty}\frac{(-1)^r x^{n+2r}}{2^{2r}.r!(n+1)(n+2)\cdots(n+r)} = au(x)$, say ....(8)

Then putting $\rho = \rho_2 = -n$ in (7),

$$c_2 = -\frac{c_0}{(-n+2+n)(-n+2-n)} = -\frac{c_0}{2.2(1-n)} = -\frac{c_0}{2^2(n-1)}$$

$$c_4 = -\frac{c_2}{(-n+4+n)(-n+4-n)} = \frac{1}{2^2.2(n-2)}\times\frac{c_0}{2^2(n-1)} = \frac{c_0}{2^4.2!(n-1)(n-2)}$$

$$c_6 = -\frac{c_4}{(-n+6+n)(-n+6-n)} = \frac{1}{2.2\times 2(n-3)}\times\frac{c_0}{2^4.2!(n-1)(n-2)}$$

$$= \frac{c_0}{2^6 3!(n-1)(n-2)(n-3)}$$

$$c_{2r} = -\frac{c_{2r}-2}{2r(2r-2n)} = \cdots = \frac{c_0}{2^{2r}.r!(n-1)(n-2)\cdots(n-r)}$$

Thus the second series solution corresponding to $\rho = \rho_2 = -n$ is obtained from (2) on putting $c_0 = b$ as

$$y_2(x) = b\sum_{r=0}^{\infty} \frac{1}{2^{2r} r!(n-1)(n-2)\cdots(n-r)} x^{-n+2r}$$

$$= bx^{-n}\left[1 + \frac{x^2}{2^n(n-1)} + \frac{x^4}{2^4.2!(n-1)(n-2)} + \frac{x^6}{2^6.3!(n-1)\cdots(n-3)} + \cdots + \frac{x^{2r}}{2^{2r}.r!(n-1)(n-2)\cdots(n-r)} + \cdots\right] \quad \ldots(9)$$

$$= bv(x), \quad \text{say}$$

Hence the general solution of the Bessel's equation (1) can be written as

$Y = y_1(x) + y_2(x) = au(x) + bv(x)$

$$= ax^n\left[1 - \frac{x^2}{2^2(n+1)} + \frac{x^4}{2^4.2!(n+1)(n+2)} + \cdots\right] + bx^{-n}\left[1 + \frac{x^2}{2^2(n-1)} + \frac{x^4}{2^4.2!(n-1)(n-2)} + \cdots\right] \ldots(10)$$

Bessel's function: If we put the constant $a = \frac{1}{2^n\Gamma(n+1)}$ in the solution $y_1(x)$ given by (8), then it is known as the **Bessel's function of the first kind of order n** and is denoted by $J_n(x)$.

Thus $J_n(x) = \frac{1}{2^n\Gamma(n+1)}\sum_{r=0}^{\infty}\frac{(-1)^r x^{n+2r}}{2^{2r}.r!(n+1)(n+2)\cdots(n+r)}$

$$= \sum_{r=0}^{\infty}\frac{(-1)^r x^{n+2r}}{2^{n+2r} r!}\frac{1}{\{\Gamma(n+1)\}(n+1)(n+2)\cdots(n+r)}$$

$$= \sum_{r=0}^{\infty}(-1)^r\left(\frac{x}{2}\right)^{n+2r} \times \frac{1}{r!}\frac{1}{\Gamma(n+r+1)}$$

i.e. $J_n(x) = \sum_{r=0}^{\infty}\frac{(-1)^r}{r!\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}$

$$\Gamma(n+1) = n\Gamma n = n(n-1)\Gamma(n-1) = \cdots = n!$$

Also, $\Gamma n = \int_0^{\infty} e^{-x} dx$

Next, putting the constant $b = \frac{1}{2^{-n}\Gamma(-n+1)}$ in the second solution $y_2(x)$ given by (9), we get **Bessel's function of the second kind of order n** and is denoted by $J_{-n}(x)$. Thus we have

$$J_{-n}(x)=\frac{1}{2^{-n}\Gamma(-n+1)}\sum_{r=0}^{\infty}\frac{x^{-n+2r}}{2^{2r}.r!(n-1)(n-2)\cdots(n-r)}$$

$$=\sum_{r=0}^{\infty}\frac{x^{-n+2r}}{2^{-n+2r}r!}\frac{1}{\{\Gamma(-n+1)\}(n-1)(n-2)\cdots(n-r)}$$

$$=\sum_{r=0}^{\infty}\left(\frac{x}{2}\right)^{-n+2r}\times\frac{1}{r!}\frac{(-1)^r}{\{\Gamma(1-n)\}(1-n)(2-n)\cdots(r-n)}$$

or $$J_{-n}(x)=\sum_{r=0}^{\infty}\frac{(-1)^r}{r!\Gamma(r-n+1)}\left(\frac{x}{2}\right)^{-n+2r}$$

The complete solution of Bessel's equation of order n is given by

$$Y = AJ_n(x) + BJ_{-n}(x)$$

Where A and B are two arbitrary constants.

Exercise 3: Solve in series the equation

$$2x^2y'' - xy' + (x^2+1)y = 0$$

Hints: Proceeding as above, the roots of the indicial equation will be found as $\rho = 1$ and 1/2. The solution will be

$$y = Ax^{\frac{1}{2}}\left(1-\frac{1}{6}x^2+\frac{1}{168}x^4-\frac{1}{66\times168}x^6+\cdots\right)+Bx\left(1-\frac{1}{10}x^2+\frac{1}{360}x^4-\frac{1}{78\times360}x^6+\cdots\right)$$

where A and B are arbitrary constants.

Case II Roots of the indicial equation — equal

Exercise 4: Solve the equation $(x-x^2)\frac{d^2y}{dx^2}+(1-5x)\frac{dy}{dx}-4y=0$ ....(1)

in series convergent near the point $x = 0$.

Solution: Here $P_1(x)=\frac{1-5x}{x-x^2}=\frac{1-5x}{x(1-x)}, P_2(x)=-\frac{4}{x(1-x)}$.

Obviously $x = 0$ and $x = 1$ are the regular singular points of the given differential equation.

Let us take the trial solution near the point $x = 0$ in the form

$$y=(x-0)^{\rho}\sum_{r=0}^{\infty}a_r(x-0)^r=\sum_{r=0}^{\infty}a_r x^{\rho+r} \quad ....(2)$$

Then $$\frac{dy}{dx} = \sum_{r=0}^{\infty} (\rho + r)\ a_r x^{\rho+r-1}$$

$$\frac{d^2y}{dx^2} = \sum_{r=0}^{\infty} (\rho + r)(\rho + r - 1)\ a_r x^{\rho+r-2}$$

On substituting these expressions for y, y′, y″, the equation (1) becomes

$$(x - x^2)\sum_{r=0}^{\infty} (\rho + r)(\rho + r - 1)\ a_r x^{\rho+r-2} + (1 - 5x)\sum_{r=0}^{\infty} (\rho + r) a_r x^{\rho+r-1}$$
$$-4\sum_{r=0}^{\infty}\ a_r x^{\rho+r} = 0$$

or $$\sum_{r=0}^{\infty} [(\rho + r)(\rho + r - 1) + (\rho + r)\ ] a_r x^{\rho+r-1}$$
$$-\sum_{r=0}^{\infty} [(\rho + r)(\rho + r - 1) + 5(\rho + r) + 4] a_r x^{\rho+r} = 0$$

Here $(\rho + r)(\rho + r - 1) + 5(\rho + r) + 4 = (\rho + r)^2 + 4(\rho + r) + 4 = \{(\rho + r) + 2\}^2$

So the preceding equation is

$$\sum_{r=0}^{\infty} (\rho + r)^2 a_r x^{\rho+r-1} - \sum_{r=0}^{\infty} (\rho + r + 2)^2 a_r x^{\rho+r} = 0$$

Or, $$\rho^2 a_0 x^{\rho-1} + \sum_{r=1}^{\infty} (\rho + r)^2 a_r x^{\rho+r-1} - \sum_{r=1}^{\infty} (\rho + r + 2)^2 a_r x^{\rho+r} = 0 \quad ....(3)$$

Here 2nd term $= \sum_{r=1}^{\infty} (\rho + r)^2 a_r x^{\rho+r-1}$

$$= \sum_{m=0}^{\infty} (\rho + m + 1)^2 a_{m+1} x^{\rho+m}$$

put $m = r - 1$,
then $r = 1 \Rightarrow m = 0$
$r \to \infty \Rightarrow m \to \infty$

Or, 2nd term $= \sum_{r=0}^{\infty} (\rho + r + 1)^2 a_{r+1} x^{\rho+r}$

So the equation (3) becomes

$$\rho^2 a_0 x^{\rho-1} + \sum_{r=0}^{\infty} \{(\rho + r + 1)^2 a_{r+1} - (\rho + r + 2)^2 a_r\} x^{\rho+r} = 0$$

Or, $$\rho^2 a_0 x^{\rho-1} + \{(\rho + 1)^2 a_1 - (\rho + 2)^2 a_0\} x^{\rho} + \sum_{r=1}^{\infty} \{(\rho + r + 1)^2 a_{r+1} - (\rho + r + 2)^2 a_r x^{\rho+r}\} = 0 \quad ......(4)$$

The coefficient of the lowest power of x, that is of $x^{\rho-1}$ in (4) equated to zero gives the indicial equation

$$\rho^2 a_0 = 0$$

$\because a_0 \neq 0$, this gives $\rho^2 = 0$, that is, $\rho = 0,0$ ....(5) are the roots of the indicial equation .

The coefficient of x^{ρ} equated to zero, gives

$$(\rho+1)^2 a_1 - (\rho+2)^2 a_0 = 0 \text{ or } a_1 = \frac{(\rho+2)^2}{(\rho+1)^2} a_0 \qquad \text{....(6)}$$

The recurrence formula is obtained by equating to zero the coefficient of $x^{\rho+r}$,

$$(\rho+r+1)^2 a_{r+1} - (\rho+r+2)^2 a_r = 0$$

Or $$a_{r+1} = \frac{(\rho+r+2)^2}{(\rho+r+1)^2} a_r, \quad r \geq 1 \qquad \text{....(7)}$$

Putting r = 1, 2, 3 etc in (7), we get

$$a_2 = \frac{(\rho+3)^2}{(\rho+2)^2} a_1 = \frac{(\rho+3)^2}{(\rho+2)^2} \times \frac{(\rho+2)^2}{(\rho+1)^2} a_0 = \frac{(\rho+3)^2}{(\rho+1)^2} a_0$$

$$a_3 = \frac{(\rho+4)^2}{(\rho+3)^2} a_2 = \cdots = \frac{(\rho+4)^2}{(\rho+1)^2} a_0 \quad \text{etc.} \qquad \text{....(8)}$$

On substituting these expressions for the constants, the series (2) turns out to be (writing $\bar{y}$ for y)

$$\bar{y} = a_0 x^{\rho}\left[1 + \left(\frac{\rho+2}{\rho+1}\right)^2 x + \left(\frac{\rho+3}{\rho+1}\right)^2 x^2 + \left(\frac{\rho+4}{\rho+1}\right)^2 x^3 + \cdots\right] \qquad \text{....(9)}$$

On putting $\rho = 0$ here, we get a solution of the given differential equation as

$$\bar{y} = a_0\left[1 + 2^2 x + 3^2 x^2 + 4^2 x^3 + \cdots\right] \qquad \text{....(10)}$$

But this gives only one solution instead of two, since we have two equal roots $\rho = 0,0$ of the indicial equation.

Now substituting the series (9) in the LHS of equation (1) and simplifying, it can be seen that

$$\left(x-x^2\right)\frac{d^2\bar{y}}{dx^2}+(1-5x)\frac{d\bar{y}}{dx}-4\bar{y}=a_0\rho^2x^{\rho-1} \qquad ....(11)$$

Regarding $\bar{y}$ as a function of x and ρ, let us differentiate both sides of the equation (11) partially with respect to ρ to obtain, $\frac{\partial}{\partial\rho}\left[\left(x-x^2\right)\frac{d^2}{dx^2}+(1-5x)\frac{d}{dx}-4\right]\bar{y}=a_0\frac{\partial}{\partial\rho}\left[\rho^2x^{\rho-1}\right]$

$$=a_0.2\rho.x^{\rho-1}+a_0\rho^2\frac{\partial}{\partial\rho}\left(x^{\rho-1}\right)$$

$$=2a_0\rho x^{\rho-1}+a_0\rho^2x^{\rho-1}\log x$$

Put $\theta=x^{\rho-1}$

or, $\log\theta=(\rho-1)\log x$

$$\therefore \frac{1}{\theta}\frac{\partial\theta}{\partial\rho}=1.\log x$$

or, $\frac{\partial\theta}{\partial\rho}=\frac{\partial}{\partial\rho}\left(x^{\rho-1}\right)=x^{\rho-1}\log x$

As the differential operators are commutative, the last equation may be written as

$$\left[\left(x-x^2\right)\frac{d^2}{dx^2}+(1-5x)\frac{d}{dx}-4\right]\frac{\partial\bar{y}}{\partial\rho}=2a_0\rho x^{\rho-1}+a_0\rho^2x^{\rho-1}\log x \qquad ....(12)$$

or, for $\rho=0$

$$\left[\left(x-x^2\right)\frac{d^2}{dx^2}+(1-5x)\frac{d}{dx}-4\right]\left(\frac{\partial\bar{y}}{\partial\rho}\right)_{\rho=0}=0$$

Hence $\frac{\partial\bar{y}}{\partial y}$ is a second solution of the given differential equation if ρ is put equal to zero after differentiation.

Now from (9) we have

$$\frac{\partial\bar{y}}{\partial\rho}=\bar{y}\log x+a_0x^\rho\left[2\left(\frac{\rho+2}{\rho+1}\right)\times\frac{-1}{(\rho+1)^2}x+2\left(\frac{\rho+3}{\rho+1}\right)\times\frac{-2}{(\rho+1)^2}x^2\right.$$

$$\left.+2\left(\frac{\rho+4}{\rho+1}\right)\times\frac{-3}{(\rho+1)^2}x^3+\cdots\right] \qquad ....(13)$$

[In obtaining the 1st term on the R.H.S. of (13) we have used that

$\frac{\partial x^\rho}{\partial\rho}=x^\rho\log x$ which is derived as given below :

Put $\theta=x^\rho$ on $\log\theta=\rho\log x$

$$\therefore \frac{1}{\theta}\frac{\partial\theta}{\partial\rho} = 1.\log x \ , \ \frac{\partial\theta}{\partial\rho} = \theta\log x \text{; i.e. } \frac{\partial x^{\rho}}{\partial\rho} = x^{\rho}\log x \ \ ]$$

Now, putting $\rho = 0$ and $a_0 = a$ and b respectively in the two series (10) and (13) we obtain

$$\bar{y} = a\left[1 + 2^2x + 3^2x^2 + 4^2x^3 + \cdots\right] = a\,u(x), \text{ say}$$

and $$\frac{\partial\bar{y}}{\partial\rho} = bu\log x + bx^0\left[2.\frac{2}{1}\times\frac{-1}{1^2}x + 2.\frac{3}{1}\times\frac{-2}{12}x^2 + 2\times\frac{4}{1}\times\frac{-3}{1^2}\times x^2 + \cdots\right]$$

$$= bu\log x - 2b\left[1.2x + 2.3x^2 + 3.4x^3 + \cdots\right] = bv(x), \text{ say}$$

Then the complete primitive is $y = a\,u + bv$.

In general, if the indicial equation has two equal roots $\rho = \alpha, \alpha$ we get two independent solutions by substituting this value of ρ in $\bar{y}$ and $\frac{\partial\bar{y}}{\partial\rho}$.

Ex. 5 : Obtain a series solution for the Bessel's equation of order zero,

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} + xy = 0 \qquad \ldots(1)$$

Solution : Here $x = 0$ is a regular singular point of the given differential equation (1).

We assume the series solution

$$y = \sum_{r=0}^{\infty} a_r x^{\rho+r} \qquad \ldots(2)$$

So that, $\frac{dy}{dx} = \sum_{r=0}^{\infty}(\rho+r)a_r x^{\rho+r-1}$; $\frac{d^2y}{dx^2} = \sum_{r=0}^{\infty}(\rho+r)(\rho+r-1)a_r x^{\rho+r-2}$

Substituting these in equation (1) we get

$$x\sum_{r=0}^{\infty}(\rho+r)(\rho+r-1)a_r x^{\rho+r-2} + \sum_{r=0}^{\infty}(\rho+r)a_r x^{\rho+r-1} + x\sum_{r=0}^{\infty}a_r x^{\rho+r} = 0$$

Or, $\sum_{r=0}^{\infty}(\rho+r)(\rho+r-1)a_r x^{\rho+r-1} + \sum_{r=0}^{\infty}a_r(\rho+r)x^{\rho+r-1} + \sum_{r=0}^{\infty}a_r x^{\rho+r+1} = 0 \ldots(4)$ Now

$$\sum_{r=0}^{\infty}a_r x^{\rho+r+1} = \sum_{m=2}^{\infty}a_{m-2}x^{\rho+m-1} = \sum_{r=2}^{\infty}a_{r-2}x^{\rho+r-1}$$

Put $r+1 = m-1$
or, $r = m-2$
when $r = 0, m = r+2 = 2$
when $\rho \to \infty, m \to \infty$

So, equation (3) becomes

$$\sum_{r=0}^{\infty}[(\rho+r)(\rho+r-1)+(\rho+r)]a_r x^{\rho+r-1}+\sum_{r=2}^{\infty}a_{r-2}x^{\rho+r-1}=0$$

Or, $$\rho^2 a_0 x^{\rho-1}+(\rho+1)^2 a_1 x^{\rho}+\sum_{r=2}^{\infty}\left[(\rho+1)^2 a_r+a_{r-2}\right]x^{\rho+r-1}=0 \quad \ldots(4)$$

Equation the coefficient of the lowest power of x to zero, we get the indicial equation,

$$\rho^2 a_0=0 \text{ or } \rho^2=0, \quad \because a_0\neq 0$$

which has the equal roots $\rho_1=\rho_2=0$ ....(5)

Equating to zero, the coefficient of x^{ρ} in (4) we get

$$(\rho+1)^2 a_1=0$$

By virtue of (5), this requires that $a_1=0$ ....(6)

The coefficient of $x^{\rho+r-1}$ in (4), equated to zero gives the recurrence formula

$$(\rho+r)^2 a_r+a_{r-2}=0\ ,\ r\geq 2$$

or, $$a_r=-\frac{a_{r-2}}{(\rho+r)^2} \quad \ldots(7)$$

From (6) and (7), we observe that all add coefficients

$$a_1=a_3=a_5=\cdots=a_{2r+1}=0$$

Then from (7) we obtain the even coefficients as

$$a_2=-\frac{a_0}{(\rho+2)^2}$$

$$a_4=-\frac{a_2}{(\rho+4)^2}=-\frac{1}{(\rho+4)^2}\times\frac{-a_0}{(\rho+2)^2}=(-1)^2\frac{a_0}{(\rho+2)^2(\rho+4)^2}$$

$$a_6=-\frac{a_4}{(\rho+6)^2}=\cdots=(-1)^3\frac{a_0}{(\rho+2)^2(\rho+4)^2(\rho+6)^2}$$

$$a_{2r}=(-1)^r\frac{a_0}{[(\rho+2)(\rho+4)\cdots(\rho+2r)]^2}$$

Putting these expressions for the coefficients in (2) we get on writing $\overline{y}$ for y

$$\bar{y} = a_0 x^\rho \left[1 - \frac{1}{(\rho+2)^2}x^2 + \frac{1}{(\rho+2)^2(\rho+4)^2}x^4 - \frac{1}{(\rho+2)^2(\rho+4)^2(\rho+6)^2}x^6\right.$$

$$\left. + \cdots + (-1)^r \frac{1}{[(\rho+2)(\rho+4)\cdots(\rho+2r)]^2} \times x^{2r} + \cdots \right] \quad \ldots(8)$$

Putting $\rho = 0$, we get the 1st series solution (on replacing a_0 by a)

$$y_1(x) = a\left[1 - \frac{1}{2^2}x^2 + \frac{1}{2^2.4^2}x^4 - \frac{1}{2^2.4^2.6^2}x^6 + \cdots\right]$$

$$= a\sum_{r=0}^{\infty}(-1)^r \frac{1}{(n!)^2}\left(\frac{x}{2}\right)^{2r} \quad \ldots(9)$$

The quantity excluding the constant 'a' on the right hand side of (9) is called the Bessel's function of the first kind of order zero and is denoted by $J_0(x)$.

To obtain the second solution, we differentiate (8) w.r.t. ρ to obtain

$$\frac{\partial \bar{y}}{\partial \rho} = a_0 \frac{\partial x^\rho}{\partial \rho}\left[1 - \frac{1}{(\rho+2)^2}x^2 + \frac{1}{(\rho+2)^2(\rho+4)^2}x^4 - \frac{1}{(\rho+2)^2\cdots(\rho+6)^2}x^6 + \cdots\right]$$

$$+ a_0 x^\rho \left[0 + \frac{2.1}{(\rho+2)^3}x^2 + \left\{\frac{-2.1}{(\rho+2)^3(\rho+4)^2} + \frac{1}{(\rho+2)^2} \times \frac{-2}{(\rho+4)^3}\right\}x^4 + \right.$$

$$\left\{\frac{2}{(\rho+2)^3(\rho+4)^2(\rho+6)^2} + \frac{2}{(\rho+2)^2(\rho+4)^3(\rho+6)^2} + \right.$$

$$\left.\left. \frac{2}{(\rho+2)^2(\rho+4)^2(\rho+6)^3}\right\}x^6 + \cdots\right]$$

Putting here $\rho = 0$ and $a_0 = b$, we get the 2nd solution as

$$y_2(x) = \left.\frac{\partial \bar{y}}{\partial \rho}\right]_{\rho=0} = bx^0 \log x\left[1 - \frac{1}{2^2}x^2 + \frac{1}{2^2.4^2}x^4 - \frac{1}{2^2.4^2.6^2}x^6 + \cdots\right]$$

$$+ bx^0\left[\frac{2}{2^3}x^2 - \left\{\frac{2}{2^3.4^2} + \frac{2}{2^2.4^3} + \cdots\right\}x^4\right.$$

$$\left. + \left\{\frac{2}{2^3.4^2.6^2} + \frac{2}{2^2.4^3.6^2} + \frac{2}{2^2.4^2.6^3}\right\}x^6 + \cdots\right]$$

$$= bJ_0(x)\log x + b\left[\frac{1}{2^2}x^2 - \frac{1}{2^2.4^2}\left(1+\frac{1}{2}\right)x^2 + \frac{1}{2^2.4^2.6^2}\left(1+\frac{1}{2}+\frac{1}{3}\right)x^6 + \cdots\right]$$

Or, $y_2(x) = bv(x)$, say.

The complete solution of the given equation is

$$Y = y_1(x) + y_2(x) = a\,J_0(x) + bv(x).$$

Case III : Roots of the Indicial Equation differing by an integer making a coefficient of $\bar{y}$ (the assumed series) infinite.

Ex 6 : Let us consider Bessel's equation of order unity,

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2-1)y = 0 \qquad \text{....(1)}$$

Here x = 0 is a regular singular point. So, we assume the series solution about x = 0 as

$$y = x^\rho\sum_{r=0}^{\infty} a_r x^r = \sum_{r=0}^{\infty} a_r x^{\rho+r} \quad \text{,where } a_0 \neq 0 \qquad \text{....(2)}$$

So that $\frac{dy}{dx} = \sum_{r=0}^{\infty}(\rho+r)a_r x^{(\rho+r-1)}$ $\qquad \frac{d^2y}{dx^2} = \sum_{r=0}^{\infty}(\rho+r)(\rho+r-1)a_r x^{(\rho+r-2)}$

Substituting these expressions in (1) we get

$$\sum_{r=0}^{\infty}(\rho+r)(\rho+r-1)a_r x^{\rho+r} + \sum_{r=0}^{\infty}(\rho+r)a_r x^{\rho+r} + \sum_{r=0}^{\infty}a_r x^{\rho+r+2} - \sum_{r=0}^{\infty}a_r x^{\rho+r} = 0$$

$$\text{....(3)}$$

Now $\sum_{r=0}^{\infty}a_r x^{\rho+r+2} = \sum_{m=2}^{\infty}a_{m-2}x^{\rho+m}$ $\qquad$ putting, $r+2=m$, $\quad r=0 \Rightarrow m-2$

$$= \sum_{r=2}^{\infty}a_{r-2}x^{\rho+r}$$

Then (3) becomes $\quad \sum_{r=0}^{\infty}[(\rho+r)(\rho+r-1)+(\rho+r)-1]a_r x^{\rho+r} + \sum_{r=2}^{\infty}a_{r-2}x^{\rho+r} = 0$

which can be rearranged to

$$(\rho^2-1)a_0x^\rho + \{(\rho+1)^2-1\}a_1x^{\rho+1} + \sum_{r=2}^{\infty}\left[\{(\rho+r)^2-1\}a_r + a_{r-2}\right]x^{\rho+r} = 0 \quad \text{....(4)}$$

The indicial equation is

$$a_0(\rho^2-1)=0, \quad \text{giving} \quad \rho=\pm 1, \quad a_0\neq 0 \qquad \ldots.(5)$$

Coefficient of $x^{\rho+1}$ gives $\{(\rho+1)^2-1\}a_1=0$ this yields $a_1=0$ ....(6)

The recurrence formula is

$$\{(\rho+r)^2-1\}a_r+a_{r-2}=0$$

$$\text{Or,} \quad a_r=-\frac{a_{r-2}}{(\rho+r)^2-1} \qquad \ldots.(7)$$

From (6) & (7) we conclude that

$$a_1=a_3=a_5=\cdots=a_{2r+1}=0$$

Putting r = 2, 4, 6 etc. in (7) we get

$$a_2=-\frac{a_0}{(\rho+2)^2-1}=-\frac{a_0}{(\rho+1)(\rho+3)}$$

$$a_4=-\frac{a_2}{(\rho+4)^2-1}=\cdots=(-1)^2\frac{a_0}{(\rho+1)(\rho+3)^2(\rho+5)}$$

$$a_6=-\frac{a_4}{(\rho+6)^2-1}=(-1)^3\frac{a_0}{(\rho+1)(\rho+3)^2(\rho+5)^2(\rho+7)}, \text{ etc.}$$

$$y=a_0x^\rho\left[1-\frac{1}{(\rho+1)(\rho+3)}x^2+\frac{1}{(\rho+1)(\rho+3)^2(\rho+5)}x^4\right.$$
$$\left.-\frac{1}{(\rho+1)(\rho+3)^2(\rho+5)^2(\rho+7)}x^6+\cdots\right] \qquad \ldots.(8)$$

we have $\rho=+1$ or -1. But the coefficient of x^2, x^4 etc. in the series (8) becomes infinite if we put $\rho=-1$, because of the factor $\frac{1}{\rho+1}$. To avoid this difficulty, the arbitrary constant a_0 is replaced by $(\rho+1)k$ where k is an arbitrary constant. The equation (8) can be written as

$$\overline{y}(x)=kx^\rho\left[(\rho+1)-\frac{1}{\rho+3}x^2+\frac{1}{(\rho+3)^2(\rho+5)}x^4-\right.$$
$$\left.-\frac{1}{(\rho+3)^2(\rho+5)^2(\rho+7)}x^6+\cdots\right] \qquad \ldots\ldots(9)$$

Again substituting this series in the given equation (1) we get

$$x^2\frac{d^2\bar{y}}{dx^2}+x\frac{d\bar{y}}{dx}+(x^2-1)\bar{y}=a_0(\rho^2-1)x^\rho=k(\rho+1)^2(\rho-1)x^\rho \quad ....(10)$$

The occurrence of the squared factor $(\rho+1)^2$ on the R.H.S. of (10) shows that $\frac{\partial\bar{y}}{\partial\rho}$ as well as $\bar{y}$ satisfy the differential equation (1) when $\rho = -1$. Also on putting $\rho = 1$ in (9) we obtain a solution. So apparently there are three solutions of the second order differential equation.

These solutions are, for $\rho = -1$

$$y_1(x)=kx^{-1}\left[0-\frac{1}{2^2}x^2+\frac{1}{2^2.4}x^4-\frac{1}{2^2.4^2.6}x^6+\cdots\right]$$

$$=k\,u,\ \text{say} \quad ....(11)$$

For $\left.\frac{\partial\bar{y}}{\partial\rho}\right]_{\rho=-1}$,

$$y_2(x)=ku\log x+kx^{-1}\left[1+\frac{1}{(\rho+3)^2}x^2+\left\{\frac{-2}{(\rho+3)^3}\times\frac{1}{(\rho+5)}-\frac{1}{(\rho+3)^2}\times\frac{1}{(\rho+5)^2}\right\}x^4\right.$$

$$\left.-\left\{\frac{-2}{(\rho+3)^2(\rho+5)^2(\rho+7)}-\frac{2}{(\rho+3)^2(\rho+5)^3(\rho+7)}-\frac{1}{(\rho+3)^2(\rho+5)^2(\rho+7)^2}\right\}x^6+\cdots\right]_{\rho=-1}$$

or, $$y_2(x)=ku\log x+kx^{-1}\left[1+\frac{1}{2^2}x^2-\frac{1}{2^2.4}\left(\frac{2}{2}+\frac{1}{4}\right)x^4+\frac{1}{2^2.4^2.6}\left(\frac{2}{2}+\frac{2}{4}+\frac{1}{6}\right)x^6+\cdots\right]$$

$$=k\,v,\ \text{say} \quad .(12)$$

And for $\rho = 1$,

$$y_3(x)=\bar{y}(x)]_{\rho=1}=kx\left[2-\frac{1}{4}x^2+\frac{1}{4^2.6}x^4-\frac{1}{4^2.6^2.8}x^6+\cdots\right]$$

$$=k\,w,\ \text{say} \quad ....(13)$$

From (11) and (13) we observe that $w = -4u$.

So we have found only two linearly independent solutions. The complete solution is

$$y = y_1(x) + y_2(x) = au + bv.$$

In general, if the indicial equation has two roots α and β (say α > β) differing by an integer and if some of the coefficients of $\bar{y}(x)$ become infinite when ρ = β, the form of the assumed series solution $\bar{y}$ is modified by replacing a_0 by k (ρ - β). Then we get two independent solutions by putting ρ = β in the modified form of $\bar{y}$ and $\frac{\partial \bar{y}}{\partial \rho}$. The result of putting ρ = α merely gives a numerical multiple of the solution corresponding to ρ = β.

Case IV : Roots of the Indicial Equation differing by an integer making a coefficient of $\bar{y}$ (the assumed series) indeterminate.

Ex 7 : Consider the equation $\quad (1-x^2)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + y = 0 \quad ....(1)$

Obviously, $x = \pm 1$ are the regular singular points of this differential equation. Instead of seeking a series solution about one of these regular singular points, let us seek a series solution about the ordinary point x = 0 in the form

$$y = x^\rho \sum_{r=0}^{\infty} a_r x^r = \sum_{r=0}^{\infty} a_r x^{\rho+r} \ ; a_0 \neq 0 \quad ....(2)$$

so that $\quad \frac{dy}{dx} = \sum_{r=0}^{\infty} (\rho+r) a_r x^{\rho+r-1} \quad$ and $\quad \frac{d^2y}{dx^2} = \sum_{r=0}^{\infty} (\rho+r)(\rho+r-1) a_r x^{\rho+r-2}$

On substituting these the equation (1) can be reduced to

$$\sum_{r=0}^{\infty} [-(\rho+r)(\rho+r-1) + 2(\rho+r) + 1] a_r x^{\rho+r} + \sum_{r=0}^{\infty} (\rho+r)(\rho+r-1) a_r x^{\rho+r-2} = 0$$

Or, $\rho(\rho-1)a_0 x^{\rho-2} + (\rho+1)\rho a_1 x^{\rho-1} + (\rho+2)(\rho+1)a_2 x^\rho + (\rho+3)(\rho+2)a_3 x^{\rho+1}$

$$+ (\rho+4)(\rho+3)a_4 x^{\rho+2} + \cdots + \{-\rho^2 + 3\rho + 1\}a_0 x^\rho + \{-(\rho+1)^2 + 3(\rho+1) + 1\}a_1 x^{\rho+1}$$

$$+ \{-(\rho+2)^2 + 3(\rho+2) + 1\}a_2 x^{\rho+2} + \cdots = 0$$

Or, $\quad \rho(\rho-1)a_0 x^{\rho-2} + (\rho+1)\rho a_1 x^{\rho-1} + [(\rho^2 + 3\rho + 2)a_2 + \{(-\rho^2 + 3\rho + 1)a_0\}]x^\rho$

$$+ [(\rho+3)(\rho+2)a_3 + \{-(\rho+1)^2 + 3(\rho+1) + 1\}a_1]x^{\rho+1} + \cdots = 0$$

The indicial equation is obtained by equating to zero the coefficient of $x^{\rho-2}$ as

$$\rho(\rho-1)a_0 = 0$$

$\because a_0 \neq 0$, we have ρ = 0 or 1 $\quad ....(3)$

The coefficient of $x^{\rho-1}$ equated to zero gives

$$a_1(\rho+1)\rho = 0 \quad ....(4)$$

Similarly, $(\rho+2)(\rho+1)a_2 + \{-\rho^2 + 3\rho + 1\}a_0 = 0 \qquad ....(5)$

$$(\rho+3)(\rho+2)a_3 + \{-(\rho+1)^2 + 3(\rho+1) + 1\}a_1 = 0 \quad ....(6) \quad \text{and so on.}$$

Now for $\rho = 0$, the coefficient of a_1 i.e. $(\rho+1)\rho$ vanishes in equation (4). This makes a_1 indeterminate instead of infinite.

If $\rho = 1$, then equation (4) requires that $a_1 = 0$. Thus if $\rho = 0$, then from equation (5), (6) etc we get $\quad 2a_2 + a_0 = 0$

and $\quad 6a_3 + \{-1+3+1\}a_1 = 0 \Rightarrow 6a_3 + 3a_1 = 0$

and $\quad 4.3a_4 + \{-4+6+1\}a_2 = 0 \Rightarrow 12a_4 + 3a_2 = 0$ and so on.

These give $a_2 = -\frac{1}{2}a_0$, $a_3 = -\frac{1}{2}a_1$, $a_4 = (-1)^2 \frac{1}{2.4}a_0$

Finally we obtain the series solution as

$$y_1(x) = y(x)]_{\rho=0} = a_0\left\{1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 + \frac{1}{80}x^4 + \cdots\right\} +$$
$$a_1\left\{x - \frac{1}{2}x^3 + \frac{1}{40}x^5 + \frac{3}{560}x^7 + \cdots\right\} \qquad .....(7)$$

This contains two arbitrary constants and so it may be taken as the complete primitive.

The series may be proved to be convergent for $|x| < 1$.

The other solutions for $\rho = 1$ is obtained as

$$y_2(x) = y(x)]_{\rho=1} = a_0 x\left[1 - \frac{1}{2}x^2 + \frac{1}{40}x^4 + \frac{3}{560}x^6 + \cdots\right]$$

This is a constant multiple of the second series in the first solution $y_1(x)$.

In general, if the indicial equation has two roots α and β ($\alpha > \beta$) differing by an integer and if one of the coefficient of y(x) becomes indeterminate when $\rho = \beta$, the complete primitive is given by putting $\rho = \beta$ in y(x), which then contains two arbitrary constants. The result of putting $\rho = \alpha$ in y(x) merely gives a numerical multiple of one of the two series contained in the first solution.

Exercises

1. Obtain a series solution of the Bessel's equation of order 2,

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-4)y=0$$

Ans : $y=y_1(x)+y_2(x)$

Where $y_1(x)=k\left[-\frac{1}{2^2.4}x^2+\frac{1}{2.2^2.4.6}x^4-\cdots\right]$

$=k\,u(x)$

and $y_2(x)=ku\log x+kx^{-2}\left[1+\frac{1}{4}x^2+\frac{1}{2^2.4^2}x^4+\cdots\right]$

2. Solve Legendre's equation of order unity,

$$(1-x^2)y''-2xy'+2y=0$$

Ans : $y=c_0\left(1-x^2-\frac{1}{3}x^4-\frac{1}{5}x^5-\cdots\right)+c_1x$.

3. Solve in series the differential equation

$(x-x^2)y''-3y'+2y=0$ about the point $x=0$. Ans : $y=a\left(1+\frac{2}{3}x+\frac{1}{3}x^2\right)+b\frac{x^4}{(1-x)^2}$

4. Examine the singularities of the differential equation

$x^3\frac{d^2y}{dx^2}+x(1-x)\frac{dy}{dx}+y=0$ and solve it in series, convergent near $x=\infty$.

Hints : Convert the independent variable x to z by the relation $x=\frac{1}{z}$

The given equation can be reduced to

$z\frac{d^2y}{dz^2}+(3-z)\frac{dy}{dz}+y=0$ Obtain the solution of this equation near $z=0$.

Ans : $y=\left(a+b\log\frac{1}{x}\right)\left(\frac{1}{x}-3\right)+b\left(x^2+3x+4-\frac{11}{3}.\frac{1}{x}+\frac{1}{8}x^2+\cdots\right)$

Simultaneous differential equations

3.1.Simultaneous equation:

In an ordinary differential equation we have two variables, one is the dependent variable say y and the other independent variable x. By the solution of the differential equation we mean that y is to be obtained as a function of x with as many constant as the order of the differential equation.

But we may have a set of ordinary differential equations involving only one independent variable and two or more dependent variables. Thus are called **simultaneous ordinary differential equations**. When these differential equations are linear these can be solved when the coefficients are constants and attempted for solution when the coefficients are variables. There are two types of simultaneous ordinary differential equations as given below: -

(i) The simultaneous linear differential equations of the type

$$f_1(D)x + f_2(D)y = T_1, \qquad \phi_1(D)x + \phi_2(D)y = T_2$$

where x, y are the dependent variables, t is the independent variables, $D = \frac{d}{dt}$, T_1, T_2 are functions of t and $f_1(D)$, $f_2(D)$, $\phi_1(D)$, $\phi_2(D)$ are differential operates with constant coefficients.

When we increase the number of dependent variables x, y to more than two, say n, then we shall require n simultaneous equation to solve them.

(ii) We may have also simultaneous equation of the form

$$P_1dx + Q_1dy + R_1dz = 0 \qquad \text{and} \qquad P_2dx + Q_2dy + R_2dz = 0$$

$P_1, P_2 \ldots\ldots R_2$ are functions of x, y and z.

These equations involve three variables x, y and z. The method of solution of these equations can be applied to equations involving any number of variables.

Solving the above equations simultaneously, we get

$$\frac{dx}{Q_1R_2 - Q_2R_1} = \frac{dy}{R_1P_2 - R_2P_1} = \frac{dz}{P_1Q_2 - P_2Q_1} \text{ which is of the form}$$

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

where P, Q, R are functions of x, y and z. We shall first discuss the methods of solving the category I of simultaneous equations and then the methods of solving equations of the category II.

3.2 Methods of solving simultaneous linear diff. equations with constant coefficient:

First method: Let us consider the simultaneous differential equations

$$f_1(D)\,x + f_2(D)y = T_1 ....(1)$$

$$\phi_1(D)x + \phi_2(D)y = T_2 ....(2)$$

Where t is the independent variable and other symbols are already defined.

We first eliminate one of the dependent variables, say y from the above two equations. For this purpose, we operate upon equation (1) by $\phi_2(D)$ and equation (2) by $f_2(D)$ to obtain,

$$f_1(D)\phi_2(D)x + f_2(D)\phi_2(D)y = \phi_2(D)T$$

and $$\phi_1(D)f_2(D)x + \phi_2(D)f_2(D)y = f_2(D)\,T$$

By subtraction, $$[f_1(D)\phi_2(D) - \phi_1(D)f_2(D)]x = [\phi_2(D) - f_2(D)]T$$

Which is the form $$F(D)x = T' ....(3)$$

This is a linear ordinary differential equation in x as dependent variable and t as independent variable. This equation can be solved for x in terms of t and as many arbitrary constant as the order of the equation (3). Then putting this expression for x in equation (1) or (2) we get an equation in y which can be solved for y.

Second Method: Method of differentiation

Sometimes x or y can be conveniently eliminated if we differentiate equation (1) or (2) or both. The resulting equation after eliminating one dependent variable (x or y) are solved to give the value of another dependent variable. Then the value of the other variable can be found.

Note: Number of arbitrary constants:

The number of arbitrary constants in the general solutions of the equations (1) and (2) is equal to the degree of D in

$$\Delta = \begin{vmatrix} f_1(D) & f_2(D) \\ \phi_1(D) & \phi_2(D) \end{vmatrix} \qquad \text{if} \quad \Delta \neq 0$$

In case $\Delta = 0$, then the system is dependent.

3.3 Worked examples

The methods of solution of this type of simultaneous equations will be made clear through the following examples.

Ex1 Solve the simultaneous equations

$$\frac{d^2x}{dt^2} - 3x - 4y = 0, \qquad \frac{d^2y}{dt^2} + x + y = 0$$

Solution: - Writing D for $\frac{d}{dt}$, the given equations can be written as

$$(D^2-3)x - 4y = 0 \ ....(1)$$

And $\quad x + (D^2+1)y = 0....(2)$

Eliminating y between these equations, we get

$$(D^2-3)(D^2+1)x + 4x = 0$$

or, $\quad (D^4-2D^2+1)x = 0$

or, $\quad (D^2-1)^2 x = 0 \qquad \therefore D = -1, -1, +1, +1$

The solution of the last equation is

$$x = (c_1 + c_2t)e^{-t} + (c_3 + c_4t)e^t \qquad ...(3)$$

Where c_1, c_2, c_3, c_4 are arbitrary constants.

Now $Dx = \frac{dx}{dt} = -(c_1 + c_2t)e^{-t} + c_2e^{-t} + (c_3 + c_4t)e^t + c_4e^t$

$$D^2x = \frac{d^2x}{dt^2} = (c_1 + c_2t)e^{-t} - c_2e^{-t} - c_2e^{-t} + (c_3 + c_4t)e^t + c_4e^t + c_4e^t$$

$$= (c_1 + c_2t)e^{-t} - 2c_2e^{-t} + (c_3 + c_4t)e^t + 2c_4e^t$$

Then from equation (1) we get $\quad 4y = (D^2-3)x = D^2x - 3x$

On substituting the expressions for x and D^2x from above and after simplification, this leads to

$$y = ½ [(c_4 - c_3 - c_4t)e^t - (c_1 + c_2t + c_2)e^{-t}] \qquad (4)$$

The solution of the given simultaneous equations are given jointly by (3) and (4).

Example 2 Solve the simultaneous equations

$$(D+1)x + (2D+7)y = e^t+2 \qquad ...(1)$$

$$-2x + (D+3)y \quad = e^t-1 \qquad ...(2)$$

Solution : Multiplying the equation (1) by 2 and operating the equation (2) by (D+1) we get

$$2(D+1)x + 2(2D+7)y = 2e^t+4$$

$$-2(D+1)x + (D+1)(D+3)y = (D+1)(e^t-1) = e^t + e^t - 1 = 2e^t - 1$$

Adding,

$$[2(2D+7) + (D+1)(D+3)]y = 4e^t + 3$$

$$\text{or } [D^2+8D+17]y = 4e^t+3 \qquad \ldots(3)$$

The auxilary equation is

$$D^2+8D+17=0$$

Its roots are $\quad D = \dfrac{-8 \pm \sqrt{64-4\cdot 1\cdot 17}}{2} = -4 \pm i$

$\therefore$ C.F. of equation (3) is $e^{-4t}(c_1e^{it} + c_2e^{-it})$ $\quad$ or $\quad$ C.F. $= a\, e^{-4t}\sin(t+b)$

where a and b are two arbitrary constants

$$\text{P.I.} = \frac{1}{D^2+8D+17}(4e^t+3) = \frac{4e^t}{1^2+8\cdot 1+17} + \frac{1}{17}\left(1+\frac{8}{17}D+\frac{1}{17}D^2\right)^{-1} 3$$

$$= \frac{4e^t}{26} + \frac{1}{17}\left(1-\frac{8}{17}D-\frac{1}{17}D^2\ldots\right)3 = \frac{2}{13}e^t + \frac{3}{17}$$

$$\therefore y = a\, e^{-4t}\sin(t+b) + \frac{2}{13}e^t + \frac{3}{17} \qquad \ldots(4)$$

The solution for x is obtained from equation (2) as

$$2x = (D+3)y - e^t + 1 = Dy + 3y - e^t + 1$$

$$= a\{-4e^{-4t}\sin(t+b) + e^{-4t}\cos(t+b)\} + \frac{2}{13}e^t + \{3ae^{-4t}\sin(t+b) + \frac{6}{13}e^t + \frac{9}{17}\} - e^t + 1$$

or, $\quad x = \dfrac{1}{2}\, ae^{-4t}\{\cos(t+b) - \sin(t+b)\} - \dfrac{5}{26}e^t + \dfrac{13}{17} \qquad \ldots(5)$

Equations (4) and (5) give the solution of the given equations.

Example 3: Solve the system of differential equations

$$(D-3)x + 2(D+2)y = 2\sin t \qquad (1)$$

$$2(D+1)x + (D-1)y = \cos t \qquad (2)$$

Solution : Operating upon equation (1) by (D-1) and upon (2) by 2(D+2), where $D = \dfrac{d}{dt}$, we get

$$(D-1)(D-3)x + 2(D-1)(D+2)y = 2(D-1)\sin t = 2\cos t - 2\sin t \quad ...(3)$$

$$4(D+2)(D+1)x + 2(D+2)(D-1)y = 2(D+2)\cos t = -2\sin t + 4\cos t ...(4)$$

Now (4) - (3)

$$[4(D^2 + 3D + 2) - (D^2 - 4D + 3)]x = 2\cos t$$

$$\text{or, } (3D+1)(D+5)x = 2\cos t \quad ...(5)$$

The auxiliary equation is

$$(3D+1)(D+5) = 0 \qquad \therefore D = -\frac{1}{3}, -5$$

The C.F. of equation (5) is $= C_1 e^{-\frac{1}{3}t} + C_2 e^{-5t}$

$$P.I = \frac{1}{3D^2 + 16D + 5} \times 2\cos t = \frac{2}{3(-1^2) + 16D + 5}\cos t = \frac{2}{16D + 2}\cos t$$

$$= \frac{(8D-1)}{(8D)^2 - 1}\cos t = \frac{(8D-1)}{64(-1)^2 - 1}\cos t = -\frac{1}{65}(8D-1)\cos t$$

$$= -\frac{1}{65}[-8\sin t - \cos t] = \frac{1}{65}[8\sin t + \cos t]$$

$$\therefore x = C_1 e^{-\frac{1}{3}t} + C_2 e^{-5t} + \frac{1}{65}[8\sin t + \cos t] \quad ...(6)$$

Then from equation (2), we get

$$(D-1)y = \cos t - 2(D+1)x = \cos t - 2[-\frac{1}{3}C_1 e^{-\frac{1}{3}t} - 5C_2 e^{-5t} + \frac{1}{65}[8\cos t - \sin t]]$$

$$-2C_1 e^{-\frac{1}{3}t} - 2C_2 e^{-5t} - \frac{2}{65}[8\sin t + \cos t]$$

$$\text{or,} \quad \frac{dy}{dt} - y = -\frac{4}{3} C_1 e^{-\frac{1}{3}t} + 8C_2 e^{-5t} + \frac{1}{65}[47\cos t - 14\sin t]$$

$$\text{Its I.F.} = e^{-\int dt} = e^{-t}$$

$$\therefore ye^{-t} = \int \left[8C_2 e^{-5t} e^{-t} - \frac{4}{3} C_1 e^{-\frac{1}{3}t} e^{-t} + \frac{1}{65}(47\cos t - 14\sin t)e^{-t} \right] dt + C_3$$

$$= \int 8C_2 e^{-6t} dt - \frac{4}{3} C_1 \int e^{-\frac{4}{3}t} dt + \frac{1}{65}\int (47\cos t - 14\sin t)e^{-t} dt + C_3$$

$$=-\frac{8}{6}C_2e^{-6t}+C_1e^{-\frac{4}{3}t}+\frac{61\sin t-33\cos t}{130}e^{-t}+C_3$$

Hence

$$y=-\frac{4}{3}C_2e^{-5t}+C_1e^{-\frac{1}{3}t}+\frac{61\sin t-33\cos t}{130}+C_3e^{t} \quad ...(7)$$

Since the degree of D in (5) of the given simultaneous equations is 2, there should be only two arbitrary constants in the solution. So, one of the arbitrary constants C_1, C_2, C_3 must be zero or a multiple of C_1, C_2, or a linear combination of C_1 and C_2. Here, on substituting the expressions (6) and (7) for x and y in equation (1), it can be seen that

$$C_3=0$$

Hence the general solution is

$$x=C_1e^{-\frac{1}{3}t}+C_2e^{-5t}+\frac{1}{65}[8\sin t+\cos t]$$

$$y=-\frac{4}{3}C_2e^{-5t}+C_1e^{-\frac{1}{3}t}+\frac{61\sin t-33\cos t}{130}$$

Example 4: Solve the simultaneous equations

$$\frac{d^2x}{dt^2}+m^2y=0, \qquad \frac{d^2y}{dt^2}-m^2x=0$$

Solution : Writing D for $\frac{d}{dt}$, the equations can be written as

$$D^2x+m^2y=0 \quad ...(1)$$

$$D^2y-m^2x=0 \quad ...(2)$$

From the first equation we obtain

$D^4x=-m^2D^2y=-m^2(m^2x)$, using the second equation

$$\therefore (D^4+m^4)x=0 \quad ...(3)$$

The auxiliary equation is

$$D^4+m^4=0$$

or $\quad (D^2+m^2)^2-2D^2m^2=0$

or, $\quad (D^2+m^2-\sqrt{2}Dm)(D^2+m^2+\sqrt{2}Dm)=0$

$\therefore$ Either $D^2+m^2-\sqrt{2}Dm=0$ or $D^2+m^2+\sqrt{2}Dm=0$

$$\text{or, } D=\frac{\sqrt{2}m\pm\sqrt{2m^2-4.1.m^2}}{2.1} \qquad \text{or, } D=\frac{-\sqrt{2}m\pm\sqrt{2m^2-4.1.m^2}}{2.1}$$

$$=\frac{\sqrt{2}m\pm\sqrt{-2m^2}}{2} \qquad =\frac{-\sqrt{2}m\pm\sqrt{-2m^2}}{2}$$

$$=\frac{m\pm im}{\sqrt{2}} \qquad =\frac{-m\pm im}{\sqrt{2}}$$

Thus $D=\dfrac{m\pm im}{\sqrt{2}},\ \dfrac{-m\pm im}{\sqrt{2}}$

Hence the solution of equation (3) is

$$x=e^{\frac{m}{\sqrt{2}}t}\left(C_1e^{\frac{imt}{\sqrt{2}}}+C_2e^{-\frac{imt}{\sqrt{2}}}\right)+e^{\frac{-m}{\sqrt{2}}t}\left(C_3e^{\frac{imt}{\sqrt{2}}}+C_4e^{-\frac{imt}{\sqrt{2}}}\right)$$

Now $$C_1e^{\frac{imt}{\sqrt{2}}}+C_2e^{-\frac{imt}{\sqrt{2}}}=C_1\left(\cos\frac{m}{\sqrt{2}}t+i\sin\frac{m}{\sqrt{2}}t\right)+C_2\left(\cos\frac{m}{\sqrt{2}}t-i\sin\frac{m}{\sqrt{2}}t\right)$$

$$=(C_1+C_2)\cos\frac{m}{\sqrt{2}}t+i(C_1-C_2)\sin\frac{m}{\sqrt{2}}t$$

| Put $C_1+C_2=a\cos b$
| $i(C_2-C_1)=a\sin b$

$$=a\cos b\cos\frac{m}{\sqrt{2}}t-a\sin b\sin\frac{m}{\sqrt{2}}t$$

$$=a\cos\left(\frac{m}{\sqrt{2}}t+b\right)$$

Similarly $$C_3e^{i\frac{m}{\sqrt{2}}t}+C_4e^{-i\frac{m}{\sqrt{2}}t}=c\cos\left(\frac{m}{\sqrt{2}}t+d\right)$$

where a, b, c and d are arbitrary constant.

$$\therefore x=ae^{\frac{m}{\sqrt{2}}t}\cos\left(\frac{m}{\sqrt{2}}t+b\right)+ce^{-\frac{m}{\sqrt{2}}t}\cos\left(\frac{m}{\sqrt{2}}t+d\right) \qquad \ldots(4)$$

Now, $$\frac{dx}{dt}=a\frac{m}{\sqrt{2}}e^{\frac{m}{\sqrt{2}}t}\cos\left(\frac{m}{\sqrt{2}}t+b\right)+ae^{\frac{m}{\sqrt{2}}t}\times\frac{-m}{\sqrt{2}}\sin\left(\frac{m}{\sqrt{2}}t+b\right)$$

$$-c\frac{m}{\sqrt{2}}e^{-\frac{m}{\sqrt{2}}t}\cos\left(\frac{m}{\sqrt{2}}t+d\right)+ce^{-\frac{m}{\sqrt{2}}t}\times\frac{-m}{\sqrt{2}}\sin\left(\frac{m}{\sqrt{2}}t+d\right)$$

$$=a\frac{m}{\sqrt{2}}e^{\frac{m}{\sqrt{2}}t}\left[\cos\left(\frac{m}{\sqrt{2}}t+b\right)-\sin\left(\frac{m}{\sqrt{2}}t+b\right)\right]$$

$$-c\frac{m}{\sqrt{2}}e^{-\frac{m}{\sqrt{2}}t}\left[\cos\left(\frac{m}{\sqrt{2}}t+d\right)+\sin\left(\frac{m}{\sqrt{2}}t+d\right)\right]$$

Differentiating this equation again with respect to t and simplifying we get

$$\frac{d^2x}{dt^2}=a\frac{m^2}{2}e^{\frac{m}{\sqrt{2}}t}\left[-\sin\left(\frac{m}{\sqrt{2}}t+b\right)\right]+Cm^2e^{-\frac{m}{\sqrt{2}}t}\sin\left(\frac{m}{\sqrt{2}}t+d\right)$$

Then from equation (1) we get $y=-\frac{1}{m^2}D^2x$

$$=ae^{\frac{m}{\sqrt{2}}t}\sin\left(\frac{m}{\sqrt{2}}t+b\right)-ce^{-\frac{m}{\sqrt{2}}t}\sin\left(\frac{m}{\sqrt{2}}t+d\right)\quad\ldots(5)$$

The expressions of (4) and (5) give the solution of the given simultaneous equations.

Example 5: Solve the system

$$(D^2-2)x-3y=e^{2y}\qquad\ldots(1)$$

$$(D^2+2)y+x=0\qquad\ldots(2)$$

where D denotes $\frac{d}{dt}$ and find the particular solution satisfying the conditions:

$x=y=1,\ Dx=Dy=0$ when $t=0$

Solution : Eliminating y from the equation (1) and (2), one obtains $(D^4-1)x=6e^{2t}$

The solution of this equation is $x=C_1e^t+C_2e^{-t}+A\cos t+B\sin t+\frac{2}{5}e^{2t}$ ...(3)

where C_1, C_2, A and B are arbitrary constants.

Then from equation (1), solution for y is obtained as $3y=(D^2-2)x-e^{2t}$

or $y=-\frac{1}{3}C_1e^t-\frac{1}{3}C_2e^{-t}-A\cos t-B\sin t-\frac{1}{15}e^{2t}$ ...(4)

Equations (3) and (4) give the general solution of the given system of equations.

Particular solution :-

When $t = 0, x = y = 1$ $\quad \therefore 1 = C_1 + C_2 + A + 0 + \frac{2}{5}$ ...(5)

$$\& \quad 1 = -\frac{1}{3} C_1 - \frac{1}{3} C_2 - A + 0 - \frac{1}{15}$$

$$\text{or} \quad 3 = -C_1 - C_2 - 3A - \frac{1}{5} \qquad \text{...(6)}$$

Adding (5) and (6), we get $\quad 4 = -2A + \frac{1}{5} \Rightarrow A = -\frac{19}{10}$ ...(7)

Again from (5) we get $\quad 3 = 3C_1 + 3C_2 + 3A + \frac{6}{5}$ ...(8)

$$\text{and} \quad 3 = -C_1 - C_2 - 3A - \frac{1}{5} \qquad \text{from (6)}$$

$$\text{Adding} \quad 6 = 2C_1 + 2C_2 + 0 + 1$$

$$\text{Or} \qquad C_1 + C_2 = \frac{6-1}{5} = \frac{5}{2} \Rightarrow C_2 = \frac{5}{2} - C_1 \qquad \text{...(9)}$$

Again, $\quad Dy = -\frac{1}{3} C_1 e^t + \frac{1}{3} C_2 e^{-t} + A\sin t - B\cos t - \frac{2}{15} e^{2t}$

$$\& \quad Dx = C_1 e^t - C_2 e^{-t} - A\sin t + B\cos t + \frac{4}{5} e^{2t}$$

Given that $Dx = Dy = 0$ at $t = 0$

$$\therefore 0 = -\frac{1}{3} C_1 + \frac{1}{3} C_2 + 0 - B - \frac{2}{15} \qquad \text{...(i)}$$

$$\& \; 0 = C_1 - C_2 - 0 + B + \frac{4}{5} \qquad \text{...(ii)}$$

Adding these two we get $\quad 0 = \frac{2}{3} C_1 - \frac{2}{3} C_2 + \frac{2}{3} \quad$ or $\quad C_1 - C_2 = -1$

Already we have $\quad C_1 + C_2 = \frac{5}{2}$

Solving these two equations we get $C_1 = \frac{3}{4}, \quad C_2 = \frac{7}{4}$

(ii)-(i) $\Rightarrow$ $$0 = \frac{4}{3}C_1 - \frac{4}{3}C_2 + 2B + \left(\frac{4}{5} + \frac{2}{15}\right)$$

or, $$2B = \frac{4}{3}C_2 - \frac{4}{3}C_1 - \frac{12+2}{15} = \frac{4}{3} - \frac{14}{15} = \frac{2}{5} \quad \text{or, } B = \frac{1}{5}$$

Thus we have $C_1 = \frac{3}{4}$, $C_2 = \frac{7}{4}$, $A = -\frac{19}{10}$, $B = \frac{1}{5}$

The required particular solution is

$$x_0 = x\Big]_{t=0} = \frac{3}{4}e^t + \frac{7}{4}e^{-t} - \frac{19}{10}\cos t + \frac{1}{5}\sin t + \frac{2}{5}e^{2t}$$
$$y_0 = y\Big]_{t=0} = -\frac{1}{4}e^t - \frac{7}{12}e^{-t} + \frac{19}{10}\cos t - \frac{1}{5}\sin t - \frac{1}{15}e^{2t}$$

Example 6: Solve the following system of differential equations

$$\frac{dx}{dt} = -x + y + z \qquad (1)$$

$$\frac{dy}{dt} = x - y + z \qquad (2)$$

$$\frac{dz}{dt} = x + y - z \qquad (3)$$

Solution : Adding the three given equations, we get

$$\frac{d}{dt}(x+y+z) = (x+y+z)$$

$$\frac{d(x+y+z)}{(x+y+z)} = dt$$

Integrating $\log(x + y + z) = t$ + a constant.

Or, $$x + y + z = Ce^t \qquad \ldots(4)$$

Where C is an arbitrary constant.

From this we have

$$-x + y + z = Ce^t - 2x \qquad (5)$$

$$x - y + z = Ce^t - 2y \qquad (6)$$

$$x + y - z = Ce^t - 2z \qquad (7)$$

From (1) and (5) we have

$$\frac{dx}{dt} = -2x + Ce^t \text{ or, } \frac{dx}{dt} + 2x = Ce^t$$

$$\text{I.F.} = e^{\int 2dt} = e^{2t}$$

$$\therefore xe^{2t} = C\int e^t.e^{2t}dt + \text{constant} = C\int e^{3t}dt + A$$

$$= C\frac{e^{3t}}{3} + A, \qquad \text{A is a constant of integration.}$$

$$\therefore x = \frac{C}{3}e^t + Ae^{-2t} \qquad \ldots(8)$$

From (2) and (6), we get $\frac{dy}{dt} = Ce^t - 2y$ or, $\frac{dy}{dt} + 2y = Ce^t$

The solution is $ye^{2t} = C\int e^t.e^{2t}dt + B$

$$= C\frac{e^{3t}}{3} + B \qquad \text{B is another constant.}$$

$$\therefore y = \frac{C}{3}e^t + Be^{-2t} \qquad \ldots(9)$$

Then from (4), (8) and (9), we have

$$z = Ce^t - x - y = Ce^t - \frac{C}{3}e^t + Ae^{-2t} - \frac{C}{3}e^t - Be^{-2t}$$

$$= \frac{C}{3}e^t - (A+B)e^{-2t} \qquad \ldots(10)$$

Hence (8), (9) and (10) give the solution of the given system of equations.

§ 3.4 Methods of solving $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

First method: we have

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{ldx + mdy + ndz}{lP + mQ + nR}$$

If l, m, n can be chosen such that $lP + mQ + nR=0$,

then we get $ldx + mdy + ndz=0$.

If it is an exact differential, say du, then u = a, a being a constant, is one part of the complete solution.

Similarly, if we can find l′, m′, n′ such that

$$l'P + m'Q + n'R = 0 \quad \text{we get} \quad l'dx + m'dy + n'dz=0$$

This gives another equation on integration.

The two equations so obtained from the complete solution of the given simultaneous equations.

Second Method: The equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

Taking any two members, say $\frac{dx}{P} = \frac{dy}{Q}$, if we find that this equation does not involve the variable z, it can be solved.

Next, we choose another two members, $\frac{dx}{P} = \frac{dz}{R}$. If this is integrable, we get another integral of the set of simultaneous equations. The two integrals, so obtained form the complete solution.

§3.5 Geometrical interpretation

We know that the direction cosines of the tangent to a space curve are

$$\left(\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}\right)$$

that is, are proportional to dx, dy, dz.

Then the simultaneous equations $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ expresses the fact that the tangent to a curve at any point (x,y,z) has direction cosines proportional to (P,Q,R). If u(x,y,z) = a and v(x,y,z)=b are two integrals of the above equations then the curves are the intersection of the surfaces u = a and v = b. Since the arbitrary constants a and b both can take any values in infinite number of ways, the curves are doubly infinite in number. In particular, if P, Q, R are constants, the curves are straight lines.

§3.6 Worked examples

Example 1. Solve the simultaneous equations

$$\frac{adx}{(b-c)yz} = \frac{bdy}{(c-a)zx} = \frac{cdz}{(a-b)xy}$$

Solution : Taking ax, by, cz as multipliers we get

$$\frac{adx}{(b-c)yz}=\frac{bdy}{(c-a)zx}=\frac{cdz}{(a-b)xy}=\frac{a^2xdx+b^2ydy+c^2zdz}{a(b-c)xyz+b(c-a)xyz+c(a-b)xyz}$$

$$=\frac{a^2xdx+b^2ydy+c^2zdz}{xyz[a(b-c)+b(c-a)+c(a-b)]}$$

$$=\frac{a^2xdx+b^2ydy+c^2zdz}{0}$$

$$\therefore a^2xdx+b^2ydy+c^2zdz=0$$

Integrating, $a^2x^2+b^2y^2+c^2z^2=$ a constant C_1 ...(1)

Again choosing x, y, z as multipliers, we get

$$\frac{adx}{(b-c)yz}=\frac{bdy}{(c-a)zx}=\frac{cdz}{(a-b)xy}=\frac{axdx+bydy+czdz}{xyz[(b-c)+(c-a)+(a-b)]}$$

$$=\frac{axdx+bydy+czdz}{0}$$

$$\therefore axdx+bydy+czdz=0$$

Integrating one gets

$$ax^2+by^2+cz^2=C_2 \qquad ...(2)$$

Then (1) and (2) together form the complete solution of the given equations.

Example 2. Solve the simultaneous equations

$$\frac{dx}{mz-ny}=\frac{dy}{nx-lz}=\frac{dz}{ly-mx}$$

Also find the radius of the circle represented by these equations that passes through the point (0, -n, -m).

Solution : We have

$$\frac{dx}{mz-ny}=\frac{dy}{nx-lz}=\frac{dz}{ly-mx}=\frac{ldx+mdy+ndz}{l(mz-ny)+m(nx-lz)+n(ly-mx)}$$

$$=\frac{ldx+mdy+ndz}{0}$$

$$\therefore ldx+mdy+ndz=0$$

Integrating $lx + my + nz = C$ ...(1)

Where C is an arbitrary constant.

Again,

$$\frac{dx}{mz-ny} = \frac{dy}{nx-lz} = \frac{dz}{ly-mx} = \frac{xdx+ydy+zdz}{x(mz-ny)+y(nx-lz)+z(ly-mx)}$$

$$= \frac{xdx+ydy+zdz}{0}$$

$$\therefore xdx + ydy + zdz = 0$$

Integrating $x^2 + y^2 + z^2 = a^2$ ...(2)

Where a is an arbitrary constant.

Integrals (1) and (2) together form the complete solution of the simultaneous equations.

2nd part: Again (1) represents a plane and (2) a sphere. So, their intersection will be circles given by the complete solution :

$$x^2 + y^2 + z^2 = a^2$$

$$\& \qquad lx + my + nz = C$$

For the circle passing through the point (0, -n, m)

We have $0 + n^2 + m^2 = a^2$ ...(i)

And $l.0+m(-n)+n.m=C \Rightarrow C = 0$ ...(ii)

Thus for the circle represented by equations (1) and (2) and passing through the point (0, -n, m), the plane (1) becomes

$$lx + my + nz = 0$$

But this plane passes through the centre O(0, 0, 0) of the sphere (2).

Then, the radius of the circle of intersection of the plane and the sphere becomes equal to the radius of the sphere itself. From relation (i) we see that this radius is $a=\sqrt{m^2+n^2}$.

Example 3 : Solve $\frac{dx}{xy} = \frac{dy}{y^2} = \frac{dz}{xyz-2x^2}$

Solution : Taking the first two ratios $\frac{dx}{xy} = \frac{dy}{y^2}$ or, $\frac{dx}{x} = \frac{dy}{y}$

Integrating, $\log x - \log y = \log C_1$ or, $\frac{x}{y} = C_1$ ...(1)

Then taking the second and third ratios,

$$\frac{dy}{y^2} = \frac{dz}{xyz - 2x^2} = \frac{dz}{C_1 y.yz - 2C_1{}^2 y^2} \qquad \text{using (1)}$$

$$\text{or} \quad \frac{dy}{y^2} = \frac{dz}{C_1 y^2 (z - 2C_1)} \quad \text{or,} \quad C_1 dy = \frac{dz}{z - 2C_1}$$

Integrating $C_1 y - \log(z-2C_1) = C_2$ ...(2)

On using (1) to remove C_1 from (2) we get

$$\frac{x}{y} y\text{-}\log(z\text{-}2\frac{x}{y}) = C_2$$

$$\text{or,} \quad x\text{-} \log(z\text{-}2\frac{x}{y}) = C_2 \qquad ...(3)$$

The complete solution is given by (1) and (3).

Example 4. Find the integral curves of the simultaneous equations

$$\frac{dx}{y^2(x-y)} = \frac{dy}{x^2(y-x)} = \frac{dz}{z(x^2+y^2)}$$

Solution : Given that

$$\frac{dx}{y^2(x-y)} = \frac{dy}{-x^2(x-y)} = \frac{dz}{z(x^2+y^2)} \qquad ...(1)$$

Taking the 1st two ratios, we get

$$x^2 dx + y^2 dy = 0$$

Integrating $x^3 + y^3 = C_1$ ...(2)

Choosing 1, -1, 0 as multiplier, each ratio of (1)

$$\text{is} = \frac{dx - dy}{y^2(x-y) + x^2(x-y)} = \frac{dx - dy}{(x-y)(x^2+y^2)}$$

Equating the 3rd ratio of (1) with this,

$$\frac{dz}{z(x^2+y^2)} = \frac{d(x-y)}{(x-y)(x^2+y^2)}$$

$$\text{or, } \frac{dz}{z} = \frac{d(x-y)}{(x-y)}$$

Integrating, $\log(x-y) - \log z = \log C_2$

$$\text{Or, } {}^{(x-y)}\!\big/_{z} = C_2 \qquad \ldots(3)$$

The required integral curves are given by the surfaces (2) and (3).

Exercises 1

1. Solve the simultaneous equations

$$\frac{d^2x}{dt^2} + 4x + y = te^t$$
$$\frac{d^2y}{dt^2} + y - 2x = \sin^2 t$$

Ans. $x = a\cos(\sqrt{3}t + b) + c\cos(\sqrt{2}t + d) + \frac{1}{6}(t - \frac{1}{6})e^t - \frac{1}{12} + \frac{1}{4}\cos 2t$

$y = -a\cos(\sqrt{3}t + b) - 2c\cos(\sqrt{2}t + d) + \frac{1}{6}e^t(t - \frac{7}{6}) + \frac{1}{3}$

Where a, b, c, d are constants.

2. Solve the simultaneous equations

$$D(D-2)x - (D-1)y = 0$$

$$(2D-1)x + D^2(D-1)y = t \qquad \text{where } D = \frac{d}{dt}.$$

Ans. $x = (C_1 + C_2 t + C_3 t^2)e^t + C_4 e^{-t} - (t+2)$, $\quad y = -[(C_1 - 2C_3)t + \frac{1}{2}C_2 t^2 + \frac{1}{3} C_3 t^3]e^t - \frac{3}{2}C_4 e^t + C_5 e^t - 2$

3. Solve the simultaneous equations

$$\frac{dx}{dt} + \frac{2}{t}(x - y) = 1$$
$$\frac{dy}{dt} + \frac{1}{t}(x + 5y) = t$$

Ans. $x = \frac{C_1}{t^3} + \frac{C_2}{t^4} + \frac{t^2}{15} + \frac{3t}{10} \qquad y = -\frac{C_1}{2t^3} - \frac{C_2}{t^4} + \frac{2t^2}{15} - \frac{1}{20}t$

4. Solve the equations

$$\frac{dx}{y^2+z^2-x^2}=\frac{dy}{-2xy}=\frac{dz}{-2xz}$$

Ans. $y=C_1z$, $x^2+y^2+z^2=C_2z$

5. Solve the simultaneous equations

$$\frac{dx}{y^3x-2x^4}=\frac{dy}{2y^4-x^3y}=\frac{dz}{9z(x^3-y^3)}$$

Ans. $xyz^{\frac{1}{3}}=C_1$, $\frac{x}{y^2}-\frac{y}{x^2}=C_2$.

6. Solve $\frac{dx}{x^2-y^2-yz}=\frac{dy}{x^2-y^2-zx}=\frac{dz}{z(x-y)}$

Ans. $x-y-z=C_1$, $\frac{x^2-y^2}{z^2}=C_2$

Total Differential Equations

4.1 Definition :

An equation of the form

$$Pdx + Qdy + Rdz=0 \qquad ...(1)$$

Where P,Q,R, are functions of x,y,z is called a total differential equation.

For example, the equations

$$(3x^2y^2-e^xz)dx+2(x^3y+\sin z)dy+(y\cos z-e^x)dz=0 \qquad ...(2)$$

$$(3xz + 2y)dx + xdy + x^2dz=0 \qquad ...(3)$$

$$\text{and } dx + dy + xdz=0 \qquad ...(4)$$

are all total differential equations.

An equation of the type (1) may be directly integrated if there exists a function u(x,y,z) whose total differential du is equal to the left hand side of equation (1). The equation (1) is then said to be an exact differential equation. Again, this equation (1) may not be exact, but may be rendered so by multiplying by a factor f(x,y,z) which is called **the integrating factor**. On the other hand, this total differential equation (1) may not be integrable at all. For equation (1) to be integrable, the coefficient functions P,Q, and R must satisfy a certain condition which is called the condition of integrability of the equation.

Now, the equation (2) can be written as

$$(3x^2y^2dx + 2x^3ydy) - (e^xz\,dx+ e^x dz)+(\sin z\, dy + y \cos z\, dz) =0$$

or, $d(x^3y^2) - d(e^xz)+d(y\sin z)=0$ or, $d(x^3y^2 - e^xz + y\sin z)=0$

Integrating, $x^3y^2 - e^xz + y\sin z = C$, a constant.

This is the primitive (or integral) of the total differential equation (2). Here,

$$u(x,y,z) = x^3y^2 - e^xz + y\sin z)$$

Next, the equation (3) can be written as $3xzdx + 2ydx + xdy + x^2dz=0$

Multiplying it by x, we get $3x^2zdx + 2xydx + x^2dy + x^3dz=0$

or, $(3x^2zdx + x^3dz)+(2xydx + x^2dy) =0$ or, $d(x^3z)+d(x^2y)=0$

Integrating, $x^3z + x^2y = C$, a constant.

This is the premitive of the total differential equation (3). Here f(x,y,z)=x is the integrating factor of the equation (3).

Finally it can be seen that equation (4) can not be integrated anyway. That is, equation (4) is not integrable.

It will be seen that the equations (2) and (3) satisfy the condition of integrability to be derived in the next section, where as the equation (4) does not satisfy it.

4.2 Condition of integrability (necessary and sufficient) for integrability of the differential equation Pdx+Qdy+Rdz=0

Necessary Condition: Consider the total differential equation

$$Pdx+Qdy+Rdz=0 \qquad \ldots(1)$$ where P,Q,R, are functions of x,y,z.

Let this equation have an integral $\phi(x,y,z) = c$ ...(2)

Then the total differential $d\phi$ must be equal to Pdx+Qdy+Rdz or to it multiplied by a factor.

But we know that

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz \qquad \ldots(3)$$

Comparing this with equation (1) we get

$$\frac{\partial\phi}{\partial x}=\lambda P, \quad \frac{\partial\phi}{\partial y}=\lambda Q, \quad \frac{\partial\phi}{\partial z}=\lambda R \qquad \ldots(4)$$

where λ is a function of x,y and z.

From the first two equations of (4), we find that

$$\frac{\partial}{\partial y}(\lambda P) = \frac{\partial^2\phi}{\partial y\partial x} = \frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial y}\right) = \frac{\partial}{\partial x}(\lambda Q)$$

$$\text{or, } \lambda\frac{\partial P}{\partial y} + P\frac{\partial\lambda}{\partial y} = \lambda\frac{\partial Q}{\partial x} + Q\frac{\partial\lambda}{\partial x}$$

$$\text{or, } \lambda\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = Q\frac{\partial\lambda}{\partial x} - P\frac{\partial\lambda}{\partial y} \qquad \ldots(5)$$

$$\text{Similarly, } \lambda\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) = R\frac{\partial\lambda}{\partial y} - Q\frac{\partial\lambda}{\partial z} \qquad \ldots(6)$$

$$\text{and } \lambda\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) = P\frac{\partial\lambda}{\partial z} - R\frac{\partial\lambda}{\partial x} \qquad \ldots(7)$$

Multiplying the equations (5),(6) and (7) by R,P and Q respectively and adding we get

$$P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=0 \qquad \ldots(8)$$

This is the necessary condition for the integrability of the equation (1).

Sufficient condition:

Theorem : We shall now prove that the condition of integrability (8) for the total differential equation (1) is sufficient. That is, when this condition, (8) is satisfied by the coefficients P,Q, and R, we can always find a solution of equation (1).

Before proving this theorem we shall establish the following lemma.

Lemma: If P,Q and R satisfy the condition (8), then so also do $P_1 = \lambda P$, $Q_1=\lambda Q$, $R_1=\lambda R$, where λ is any function of x, y and z.

Proof of the lemma

$$P_1\left(\frac{\partial Q_1}{\partial z}-\frac{\partial R_1}{\partial y}\right)=\lambda P\left\{\left(\lambda\frac{\partial Q}{\partial z}+Q\frac{\partial \lambda}{\partial z}\right)-\left(\lambda\frac{\partial R}{\partial y}+R\frac{\partial \lambda}{\partial y}\right)\right\}$$

$$=\lambda^2 P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+\lambda P\left(Q\frac{\partial \lambda}{\partial z}-R\frac{\partial \lambda}{\partial y}\right)$$

Similarly,

$$Q_1\left(\frac{\partial R_1}{\partial x}-\frac{\partial P_1}{\partial z}\right)=\lambda^2 Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+\lambda Q\left(R\frac{\partial \lambda}{\partial x}-P\frac{\partial \lambda}{\partial z}\right)$$

$$R_1\left(\frac{\partial P_1}{\partial y}-\frac{\partial Q_1}{\partial x}\right)=\lambda^2 R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)+\lambda R\left(P\frac{\partial \lambda}{\partial y}-Q\frac{\partial \lambda}{\partial x}\right)$$

Adding these three results we get

$$\sum P_1\left(\frac{\partial Q_1}{\partial z}-\frac{\partial R_1}{\partial y}\right)=\lambda^2\sum P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)$$

$$+\lambda\left(PQ\frac{\partial \lambda}{\partial z}-PR\frac{\partial \lambda}{\partial y}+QR\frac{\partial \lambda}{\partial x}-QP\frac{\partial \lambda}{\partial z}+RP\frac{\partial \lambda}{\partial y}-RQ\frac{\partial \lambda}{\partial x}\right)$$

$$=\lambda^2 .0+0=0$$

Hence

$$P_1\left(\frac{\partial Q_1}{\partial z}-\frac{\partial R_1}{\partial y}\right)+Q_1\left(\frac{\partial R_1}{\partial x}-\frac{\partial P_1}{\partial z}\right)+R_1\left(\frac{\partial P_1}{\partial y}-\frac{\partial Q_1}{\partial x}\right)=0 \qquad \ldots(9)$$

Proof of the Theorem

To obtain the solution of equation (1), we assume, for the time being, z to be constant so that dz = 0. Then equation (1) becomes Pdx+Qdy = 0

Let the solution of this equation be $F(x,y,z) = a$ (10)

This gives $\frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = 0$

$$\text{So} \quad \frac{\frac{\partial F}{\partial x}}{P} = \frac{\frac{\partial F}{\partial y}}{Q} = \lambda\text{, say}$$

Now put $P_1=P\lambda$, $Q_1=Q\lambda$, $R_1=R\lambda$ and replace 'a' in (10) by f(z), giving

$$F(x,y,z)=f(z) \qquad (11)$$

$$\text{Hence} \quad \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial z}dz = \frac{df}{dz}dz$$

$$\text{Or,} \quad \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + \left(\frac{\partial F}{\partial z} - \frac{df}{dz}\right)dz = 0$$

$$\text{Or,} \quad P_1dx + Q_1dy + \left(\frac{\partial F}{\partial z} - \frac{df}{dz}\right)dz = 0 \qquad ...(12)$$

This will be identical with the equation

$$Pdx + Qdy + Rdz = 0 \qquad ...(1)$$

$$\text{If} \quad \frac{\partial F}{\partial z} - \frac{df}{dz} = \lambda R = R_1$$

$$\text{i.e.} \quad \text{If } \frac{df}{dz} = \frac{\partial F}{\partial z} - R_1 \qquad ...(13)$$

Now we shall show that the right hand side of equation (13) can be reduced to a function of F and z only by virtue of the relation (11). That is $\frac{\partial F}{\partial z} - R_1$ involves x and y only as a function of F.

It is known from differential calculus that $\frac{\partial F(x,y,z)}{\partial z} - R_1(x,y,z)$ will not involve x and y if

$$\frac{\partial F}{\partial x}\frac{\partial}{\partial y}\left\{\frac{\partial F}{\partial z} - R_1\right\} - \frac{\partial F}{\partial y}\frac{\partial}{\partial x}\left\{\frac{\partial F}{\partial z} - R_1\right\} \equiv 0 \qquad ...(14)$$

Again the lemma (9) is

$$P_1\left(\frac{\partial Q_1}{\partial z}-\frac{\partial R_1}{\partial y}\right)+Q_1\left(\frac{\partial R_1}{\partial x}-\frac{\partial P_1}{\partial z}\right)+R_1\left(\frac{\partial P_1}{\partial y}-\frac{\partial Q_1}{\partial x}\right)=0 \quad ...(9)$$

further, since the equation (12) is integrable, we have

$$P_1\left\{\frac{\partial Q_1}{\partial z}-\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial z}-\frac{df}{dz}\right)\right\}+Q_1\left\{\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial z}-\frac{df}{dz}\right)-\frac{\partial P_1}{\partial z}\right\}+\left(\frac{\partial F}{\partial z}-\frac{df}{dz}\right)\left(\frac{\partial P_1}{\partial y}-\frac{\partial Q_1}{\partial x}\right)=0$$

Subtracting this from equation (9) we get

$$P_1\frac{\partial}{\partial y}\left\{\frac{\partial F}{\partial z}-\frac{df}{dz}-R_1\right\}-Q_1\frac{\partial}{\partial x}\left\{\frac{\partial F}{\partial z}-\frac{df}{dz}-R_1\right\}-\left\{\frac{\partial F}{\partial z}-\frac{df}{dz}-R_1\right\}\left(\frac{\partial F_1}{\partial y}-\frac{\partial Q_1}{\partial x}\right)=0 \quad ...(15)$$

But $P_1=\lambda P=\frac{\partial F}{\partial x}$, $Q_1=\frac{\partial F}{\partial y}$, $\frac{\partial}{\partial x}\left(\frac{df}{dz}\right)=\frac{\partial}{\partial y}\left(\frac{df}{dz}\right)=0$

As f is a function of z alone.

$\therefore$ (15) becomes

$$\frac{\partial F}{\partial x}\frac{\partial}{\partial y}\left(\frac{dF}{dz}-R_1\right)-\frac{\partial F}{\partial y}\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial z}-R_1\right)-\left(\frac{\partial F}{\partial z}-\frac{df}{dz}-R_1\right)\left\{\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right)-\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right)\right\}=0$$

or, $$\frac{\partial F}{\partial x}\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial z}-R_1\right)-\frac{\partial F}{\partial y}\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial z}-R_1\right)=0$$

This is the same as the condition (14).

Thus, $\frac{\partial F}{\partial z}-R_1$ can be expressed as a function of F and z, say $\psi(F,z)$.

Hence, from (13), $\frac{df}{dz}=\psi(F,z)=\psi(f,z)$ using (11)

Let the solution of this equation be $F(z)=\chi(z)$

Then $F(x,y,z)=f(z)=\chi(z)$ is a solution of $Pdx+Qdy+Rdz=0$ (1)

Thus the equation (1) is proved to be integrable whenever P,Q,R satisfy the condition (8).

4.3 Geometrical interpretation of the equation

$$Pdx+Qdy+Rdz=0 \quad ...(1)$$

This differential equation expresses the fact that the tangent to a curve is perpendicular to a certain line. The direction cosines of the tangent are proportional to dx, dy, dz and the direction cosines of the line are proportional to P, Q, R.

But the simultaneous equations

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R} \qquad \ldots(2)$$

expresses that the tangent to a curve is parallel to the line (P,Q,R). We thus get two sets of curves. If two curves, one of each set, intersect, they must do so at right angles.

Now two cases arise.

If may happen that the equation (1) is integrable. This means that a family of surfaces can be found, all curves on which are perpendicular to the curves represented by the simultaneous equations (2) at all points where these curves cut the surfaces.

On the other hand, the curves represented by the simultaneous equations may not admit of such a family of orthogonal surfaces . In this case, the single equation (1) is non-integrable.

4.4 Condition of exactness

Suppose the equation Pdx+Qdy+Rdz=0 ...(1) is exact. That is it can be integrated by rearranging its terms.

Let its integral be u(x,y,z)=C Then du= Pdx+Qdy+Rdz

Also, $du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+\frac{\partial u}{\partial z}dz$. So, we have $P=\frac{\partial u}{\partial x}$, $Q=\frac{\partial u}{\partial y}$, $R=\frac{\partial u}{\partial z}$

But $\frac{\partial P}{\partial y}=\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)=\frac{\partial}{\partial x}Q$ Similarly $\frac{\partial Q}{\partial z}=\frac{\partial R}{\partial y}$ and $\frac{\partial R}{\partial x}=\frac{\partial P}{\partial z}$

These are the conditions of exactness of the equation (1).When these are satisfied the conditionof integrability

$$P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=0$$ is automatically satisfied.

4.5 Methods of Solving Pdx+Qdy+Rdz=0 ...(1)

Case I: Exact equation

When the conditions of exactness are satisfied, the equation (1) can be integrated directly by rearranging its terms and expressing the right hand side as the total differential du of a function u(x,y,z). The integral will be u(x,y,z)=C, a constant.

When these conditions for exactness are not satisfied by P,Q,R, then we have to check whether they satisfy the condition of integrability:-

$$P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=0 \qquad (2)$$

If this condition is satisfied, the equation (1) is integrable and the following cases arise:-

Case II Homogeneous Equation

If P, Q, R are homogeneous functions of x,y,z then one of these variables, say, z can be separated from the others by the substitution x=zu, y=zv, so that dx=zdu+udz and dy= zdv+vdz.

On substituting these in the given equation (1), it will be reduced to an equation involving only u and v or an equation where the variable z is separated from u and v. In both the cases the reduced equation can be integrated as an exact equation.

Case III One variable constant

If any two terms of the equation (1) equated to zero, say Pdx+Qdu=0 can be readily integrated, then we take the third variable z= constant , so that dz=0.

Thus the equation (1) becomes Pdx+Qdy=0

Let its solution be $u=Q(z)$ ...(3)

Where $\phi(z)$ is an arbitrary function of z alone and independent of x and y. Here u=u(x,y).

Then to find the solution of equation (1) completely we take total differential of equation (3) as

$$\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+\frac{d\phi(z)}{dz}dz=0 \qquad \text{...(4)}$$

Comparing equations (4) and (1) we get a relation between $\phi(z)$ and z. If the coefficient of $d\phi$ or dz in this relation involves x and y, it will be possible to remove them by using (3).

Solving this equation we obtain $\phi(z)$ and substituting its value in (3) we obtain the complete primitive of equation (1).

Case IV Method of auxiliary equations

If none of the above methods is applicable then comparing the equations (1) and (2) we get the simultaneous equations

$$\frac{dx}{\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}}=\frac{dy}{\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}}=\frac{dz}{\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}} \qquad \text{...(5)}$$

which are called auxiliary equations and can be solved like simultaneous equations.

Let u(x,y,z)=a and v(x,y,z)=b

be the two integrals of the auxiliary equations (5). Then comparing Adu+Bdv=0 with equation(1) we get the values of A and B and then the complete primitive.

It may be noted that this method is not applicable for an exact equation.

§4.6 Worked Examples

Example 1 Verify that the equation

$$(y - z)(y + z - 2x)dx + (z - x)(z+x-2y)dy+(x - y)(x + y - 2z)dz=0$$

is exact and find the solution.

Solution : Comparing this equation with $Pdx + Qdy + Rdz = 0$

We find that $P= y^2 - z^2-2x(y-z)$, $Q=z^2 - x^2-2y(z - x)$, $R= x^2 - y^2 - 2z(x - y)$

$$\frac{\partial P}{\partial y}=2y-2x, \quad \frac{\partial P}{\partial z}=-2z+2x, \quad \frac{\partial Q}{\partial z}=2z-2y,$$

$$\frac{\partial Q}{\partial x}=-2x+2y, \quad \frac{\partial R}{\partial x}=2x-2z, \quad \frac{\partial R}{\partial y}=-2y+2z$$

We see that $\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}$, $\frac{\partial Q}{\partial z}=\frac{\partial R}{\partial y}$ and $\frac{\partial R}{\partial x}=\frac{\partial P}{\partial z}$

Thus the conditions of exactness is satisfied and the given equation is exact.

The given equation can be written as

$$(y^2 - z^2-2xy+2xz)dx + (z^2 - x^2-2yz +2yx)dy + (x^2 - y^2 - 2zx + 2zy)dz=0$$

or, $(y^2dx +2xydy) -(z^2 dx +2xzdz) + (z^2 dy +2yzdz) -(x^2dy+2yxdy)$

$$+ (x^2 dz +2xzdx) - (y^2 dz + 2zydy)=0$$

or, $d(y^2x) - d(z^2x) + d(z^2y) - d(x^2y) + d(x^2z) + d(y^2z)=0$

Integrating $y^2x - z^2x + z^2y - x^2y + x^2z + y^2z = C$ where C is an arbitrary constant.

Example 2: Solve $zdz+(x-a)dx =\sqrt{\{h^2 - z^2-(x-a)^2\}}dy$

Solution : Here $P = x-a$, $Q= - \{h^2 - z^2-(x-a)^2\}^{1/2}$, $R = z$

$$\frac{\partial P}{\partial y}= 0 =\frac{\partial P}{\partial z}, \quad \frac{\partial Q}{\partial x}=-\frac{1}{2}\frac{1\times\{-2(x-a).1\}}{\{h^2 - z^2 - (x-a)^2\}^{1/2}}=\frac{(x-a)}{\{h^2 - z^2 - (x-a)^2\}^{1/2}}$$

$$\frac{\partial Q}{\partial z}=-\frac{1}{2}\frac{1\times(-2z)}{\{h^2 - z^2 - (x-a)^2\}^{1/2}}=\frac{z}{\{h^2 - z^2 - (x-a)^2\}^{1/2}}$$

$$\frac{\partial R}{\partial x}=0=\frac{\partial R}{\partial y}$$

Now, $P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)$

$$=(x-a)\left\{\frac{z}{\{h^2-z^2-(x-a)^2\}^{1/2}}-0\right\}+\{h^2-z^2-(x-a)^2\}^{1/2}(0-0)$$

$$+z\left[0-\frac{x-a}{\{h^2-z^2-(x-a)^2\}^{1/2}}\right]$$

$$=\frac{(x-a)z}{\{h^2-z^2-(x-a)^2\}^{1/2}}+0-\frac{(x-a)z}{\{h^2-z^2-(x-a)^2\}^{1/2}}=0.$$

Thus the condition of integrability is satisfied.

The given equation can be written as

$$\frac{zdz+(x-a)dx}{[h^2-\{z^2+(x-a)^2\}]^{1/2}}=dy$$

or, $$-\frac{1}{2}\frac{d[h^2-\{z^2+(x-a)^2\}]}{[h^2-\{z^2+(x-a)^2\}]^{1/2}}=dy$$

Integrating $$-\frac{1}{2}\times\frac{[h^2-\{z^2+(x-a)^2\}]^{1/2}}{1/2}=y+\cos t$$

$$\therefore [h^2-\{z^2+(x-a)^2\}]=(y-c)^2$$ is the solution.

Example 3: Solve the equation

$$(y^2+yz)dx+(zx+z^2)dy+(y^2-xy)dz=0$$

Solution : Comparing the equation with $Pdx+Qdy+Rdz=0$

We find $P=y^2+yz$, $Q=zx+z^2$, $R=y^2-xy$

$$\frac{\partial P}{\partial y}=2y+z,\ \frac{\partial P}{\partial z}=y,\ \frac{\partial Q}{\partial z}=2z+x,\ \frac{\partial Q}{\partial x}=z$$

$$\frac{\partial Q}{\partial x}=-y,\quad \frac{\partial R}{\partial y}=2y-x$$

Now, $$P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)$$

$$=(y^2+yz)\{(2z+x)-(2y-x)\} + (zx+z^2)\{-y-y\} + (y^2-xy)\{(2y+z)-z\}$$

$$=(y^2+yz)\{2x+2z-2y\} - 2y(zx+z^2) + 2y(y^2-xy)$$

$$=2xy^2 + 2y^2z - 2y^3 + 2xyz + 2yz^2 - 2zy^2 - 2yz^2 - 2xyz + 2y^3 - 2xy^2$$

$$=0.$$

The condition of integrability is satisfied.

Again the given equation is homogeneous in x,y,z.

So we take x = uz, y = vz, so that dx = udz + zdu, dy = vdz + zdv

Then the given equation becomes

$$(v^2z^2+vz^2)(udz+zdu)+(uz^2+z^2)(vdz+zdv)+(v^2z^2-uvz^2)dz=0$$

or dividing throughout by z^2,

$$[(v^2+v)u+(u+1)v+(v^2-uv)]dz+(v^2+v)zdu+(u+1)zdv=0$$

or, $(v^2+v)(u+1)dz+z[(v^2+v)du+(u+1)dv]=0$

or, $\dfrac{dz}{z}+\dfrac{du}{u+1}+\dfrac{dv}{v^2+v}=0$

or, $\dfrac{dz}{z}+\dfrac{d(u+1)}{(u+1)}+\dfrac{dv}{v}-\dfrac{d(v+1)}{(v+1)}=0$

Integrating we get

$$\log z+\log(u+1)+\log v-\log(v+1)=\text{constant}=\log C, \text{ say}$$

or, $zv(u+1)=c(v+1)$

or, $z\dfrac{y}{z}(\dfrac{x}{z}+1)=c(\dfrac{y}{z}+1)$

or, $y(x+z)=c(y+z)$ which is the primitive of the given equations.

Example 4 Solve the equation

$$(x^2y-y^3-y^2z)dx+(xy^2-x^2z-x^3)dy+(xy^2+x^2y)dz=0$$

Solution : It can be shown that this equation is integrable.

Further the equation is homogeneous in x, y & z. So we put x = uz, y= vz, so that dx=udz+zdu, dy= vdz+zdv.

Then, substituting, these and simplifying, the given equation can be reduced to

$(u^2 - v^2)(vdu-udv) - v^2 du - u^2dv = 0$

dividing it by u^2v^2 and simplifying we get

$$\frac{vdu-udv}{v^2}+\frac{udv-vdu}{u^2}-\frac{1}{v^2}dv-\frac{1}{u^2}du=0$$

$$\text{or,}\quad d\left(\frac{u}{v}\right)+d\left(\frac{v}{u}\right)+d\left(\frac{1}{v}\right)+d\left(\frac{1}{u}\right)=0$$

$$\text{Integrating,}\quad \frac{u}{v}+\frac{v}{u}+\frac{1}{v}+\frac{1}{u}=0$$

Or, $\frac{x}{y}+\frac{y}{x}+\frac{z}{x}+\frac{z}{y}=C$ which is the integral of the given equation.

Example 5. Solve $(2x^2+2xy+2xz^2+1)dx + dy + 2zdz=0$ (1)

Solution : Here $P=2x^2+2xy+2xz^2+1$, $Q=1$, $R=2z$

It can be shown that the condition of integrability is satisfied by the coefficients P,Q and R. Hence the given equation is integrable.

To integrate the equation , we treat x to be a constant for the time being, so that dx=0 and the given equation reduces to $dy + 2zdz=0$.

Integrating this, $y+z^2$ = constant with respect to y and $z = f(x)$ ...(2), say

Now differentiating (2), we obtain

$$-\frac{df(x)}{dx}dx+dy+2zdz=0 \qquad \ldots(3)$$

This equation should be identical with equation (1), if (2) is an integral of (1). Then comparing (3) with (1) we get

$$\frac{-\frac{df(x)}{dx}}{2x^2+2xy+2xz^2+1}=\frac{1}{1}=\frac{2z}{2z}$$

$$\therefore \frac{df(x)}{dx}=-(2x^2+2xy+2xz^2+1)=-2x^2-1-2x(y+z^2)$$

$$=-2x^2-1-2xf(x) \qquad \text{using (2)}$$

$$\therefore \frac{df}{dx}+2xf=-(2x^2+1)$$

Its I.F. $=e^{\int 2x dx}=e^{x^2}$

$$\therefore fe^{x^2} = -\int (2x^2+1)e^{x^2}dx + \text{constant}$$

$$= -\int 2x^2e^{x^2}dx - \int e^{x^2}dx + C \qquad \ldots(4)$$

Here $\int 2x^2e^{x^2}dx = \int x.2xdxe^{x^2} = \int x.e^{x^2}d(x^2)$

$$= xe^{x^2} - \int \frac{1}{2x}.e^{x^2}dx^2 \qquad \left|\ \frac{dx}{dx^2} = \frac{d}{dt}t^{1/2}\right.$$

$$= xe^{x^2} - \int e^{x^2}dx \qquad \left|\ = \frac{1}{2t^{1/2}} = \frac{1}{2x}\right.$$

then (4)$\Rightarrow$ $fe^{x^2} = -xe^{x^2} + \int e^{x^2}dx - \int e^{x^2}dx + C$

$$= -xe^{x^2} + C$$

or $\quad f = -x + Ce^{-x^2}$

Hence the integral of (2) becomes $y + z^2 = -x + ce^{-x^2}$ which is the complete solution of (1)

Example 6: Find f(y) if f(y)dx – zxdy – xylogydz=0 is integrable. Find the corresponding integral.

Solution : Here P=f(y), Q = -zx, R= -xylogy.

If the given equation is integrable, then P,Q,R must satisfy the following condition

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0$$

or, $f(y)[-x-(-x\log y - xy.\frac{1}{y})] - zx[-y\log y - 0] - xy\log y[f'(y) - (-z)] = 0$

or, $f(y).x\log y - xy\log y.f'(y) = 0$

$$\text{or, } \frac{f'(y)}{f(y)} = \frac{1}{y}$$

integrating, $\log f(y) = \log y + \log C$

$$\text{or, } f(y) = Cy \qquad \ldots(2)$$

Then the given equation becomes

$$cydx - zxdy - xy\log ydz = 0 \qquad \ldots(3)$$

Treating z as constant, that is dz = 0, (3) becomes

$$cydx - zxdy = 0$$

$$\text{or, } \frac{Cdx}{x} - z\frac{dy}{y} = 0$$

Integrating, by taking z as constant, we get

clogx – zlogy = constant with respect to x and y = F(z) say ...(4),

Where F(z) is an arbitrary function of z.

Differentiating (4), we get, taking now z as a variable,

$$\frac{C}{x}dx - \frac{z}{y}dy - \log ydz = \frac{dF}{dz}dz$$

$$\text{or } cydx - zxdy - [xylogy + xy\frac{dF}{dz}]dz = 0 \qquad ...(5)$$

Comparing (3) and (5) we get

$$xylogy + xy\frac{dF}{dz} = xylogy$$

or, $\frac{dF}{dz}=0$, Integrating $F = C_1$, an arbitrary constant.

Putting $F = C_1$ in (4), required solution is

$$Clogx - zlogy = C_1$$

Example 7: Solve the equation

$$3x^2dx + 3y^2dy - (x^3 + y^3 + e^{2z})dz = 0 \qquad ...(1)$$

Solution : Here $P=3x^2$, $Q= 3y^2$, $R= -(x^3 + y^3 + e^{2z})$

$$\frac{\partial P}{\partial y}=0=\frac{\partial P}{\partial z}, \frac{\partial Q}{\partial x}=0=\frac{\partial Q}{\partial z}, \frac{\partial R}{\partial y}=-3y^2, \frac{\partial R}{\partial x}=-3x^2$$

It can be shown that the condition of integrability is satisfied.

The auxiliary equations are

$$\frac{dx}{\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}} = \frac{dy}{\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}} = \frac{dz}{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}$$

$$\text{or,}\quad \frac{dx}{0+3y^2}=\frac{dy}{-3x^2-0}=\frac{dz}{0-0}$$

$$\text{or,}\quad \frac{dx}{y^2}=\frac{dy}{-x^2}=\frac{dz}{0}$$

The first two ratios give $x^2dx + y^2dy = 0$

Integrating $x^3 + y^3$ = constant with respect to x and y = u, say.

The last ratio gives dz =0 $\Rightarrow$z = constant with respect to z = v, say.

$$\text{Then}\quad Adu + Bdv = 0 \qquad \ldots(2)$$

$$\text{Gives}\quad A(3x^2\,dx + 3y^2dy) + Bdz = 0$$

Comparing this with the given equation one obtains

$$3x^2 = 3Ax^2,\ 3y^2 = 3Ay^2,\ -(x^3+y^3+e^{2z})=B$$

$$\therefore A = 1 \text{ and } B = -(u+e^{2v})$$

Hence (2) becomes $\qquad du = -(u+e^{2v})dv=0$

$$\text{or,}\quad \frac{du}{dv}-u=e^{2v} \qquad \ldots(3)$$

$$\text{Its I.F.} = e^{\int -dv} = e^{-v}$$

$$\text{Hence}\quad ue^{-v}=\int e^{2v}e^{-v}dv+C=\int e^{v}dv+C=e^{v}+C$$

$$\therefore u= Ce^{v}+e^{2v}$$

$\therefore$ the solution is $\qquad x^3 + y^3= Ce^{z}+e^{2z}$

§4.7 Non-integrable single equation

If the equation $Pdx + Qdy + Rdz = 0 \qquad \ldots(1)$

does not satisfy the condition of integrability, then this equation cannot, in general, be integrated.

In this case, the equation (1) represents a family of curves orthogonal to the family represented by the simultaneous equations

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R} \qquad \ldots(2)$$

But in this case there is no family of surfaces orthogonal to the second family of curves.

However, we can find an infinite number of curves that lie on any given surface. And satisfy the equation (1), whether that equation is integrable or not.

Suppose we are given an arbitrary equation

$$f(x,y,z)=C \quad \ldots(3)$$

in x,y,z. Then the solution of equation (1) can be determined subject to the relation (3) as follows:-

From (3) we get

$$\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy+\frac{\partial f}{\partial z}dz=0 \quad \ldots(4)$$

When the form of the relation (3) is known, then one variable and its differential can be determined in terms of the other variables and their differentials. Thus from (1), (3) and (4) one variable and differential say z and dz can be eliminated. Then we get a differential equation of the form

$$P_1dx + Q_1dy=0 \quad \ldots(5)$$

Where P_1 and Q_1 are functions of x and y only. The forms of P_1 and Q_1 depend upon (3) containing the arbitrary constant C of (3).

The equation (5) can now be solved. This solution together with the equation (3) forms a solution. For different values of f, different solutions can be obtained.

Obviously this solution represents a family of curves that lie on the surface (3) and satisfy the equation (1).

§4.8 Worked examples

Example 8 Find the curves represented by the solution of

$$ydx + (z-y)dy + xdz = 1 \quad \ldots(1)$$

which lie in the plane $2x - y - z = 1$ ...(2)

Solution : Here P = y, Q = z-y, R = x

$$\text{Hence } P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)$$

$$= y(1-0) + (z-y)(1-0) + x(1-0)$$

$$= y + z - y + x = x+z \neq 0$$

Thus the condition of integrability is not satisfied.

Now differentiating the equation (2) we get $2dx - dy - dz = 0$ ...(3)

We shall now eliminate the variable z and its differential dz from (1), (2) and (3).

For the purpose, x×(3) + (1) gives $(y+2x)dx + (z-y-x)dy = 0$

or on using (2), $(y+2x)dx + (x-2y-1)dy = 0$

or, $(ydx + xdy) + 2xdx - 2ydy - dy = 0$

Integrating, $xy + x^2 - y^2 - y = \text{constant}, C^2$, say ...(4)

Thus the curves of the family represented by equation (1) and which lie in the plane (2) are section by that plane of the infinite set of rectangular hyperbolic cylinders given by equation (4).

Example 9. Show that there is no single integral of

$$dz = 2ydx + xdy \qquad \text{...(1)}$$

Prove that the curves of this equation that lie in the plane $z = x + y$ ...(2)

lie also on surfaces of the family $(x-1)^2(2y-1)=C$ ...(3)

Solution : It can be shown that the condition of integrability is not satisfied by the equation (1).

Differentiating equation (2) we get

$$dx + dy = dz \qquad \text{...(4)}$$

To eliminate z and dz beyween (1), (2) and (4), we have by subtracting (4) from (1),

$$(2y - 1)dx + (x-1)dy = 0$$

$$\text{or} \qquad \frac{dx}{x-1} + \frac{dy}{2y-1} = 0$$

$$\text{or} \qquad \frac{d(x-1)}{(x-1)} + \frac{1}{2}\frac{d(2y-1)}{(2y-1)} = 0$$

Integrating, $2\log(x-1) + \log(2y-1) = \log C$

$$\therefore (x-1)^2(2y-1) = C \qquad \text{...(3)}$$

Thus the curves of the equation (1) that lie in the plane (2), lie also in the surface represented by the equation (3).

Exercise 2

Solve $\frac{yz}{x^2+y^2}dx - \frac{zx}{x^2+y^2}dy - \tan^{-1}\frac{y}{x}dz = 0$ **Ans.** $y/x = \tan\left(\frac{1}{cz}\right)$

Solve $(2x^3y+1)dx + x^4dy + x^2\tan z dz = 0$ **Ans.** $x^2y - \frac{1}{x} + \log\sec z = C$

Solve $yz\log z dx - zx\log z dy + xy dz = 0$ **Ans.** $x\log z = cy$

Solve $z(1-z^2)dx + zdy - (x+y+xz^2)dz = 0$ **Ans.** $x - xz^2 + y = cz$

Find f(z) such that

$[(y^2 + z^2 - x^2)/2x]dx - ydy + f(z)dz = 0$ is integrable. Hence solve it.

Ans. $xc = x^2 + y^2 + z^2$, c is arbitrary.

Solve $(yz+z^2)dx - xzdy + xydz = 0$ **Ans.** $xz = c(y+z)$

Solve $(y^2+yz+z^2)dx + (z^2+xz+x^2)dy + (x^2+xy+y^2)dz = 0$ **Ans.** $(xy+yz+zx) = c(x+y+z)$

Solve $(z+z^3)\cos x dx - (z+z^3)dy + (1-z^2)(y-\sin x)dz = 0$ **Ans.** $(\sin x - y)(z^2+1) = cz$

Solve $(1+yz)dx + (zx - x^2)dy - (1+xy)dz = 0$ **Ans.** $1+xy = c(z-x)$

Solve $z(z-y)dx + (zx - x^2)dy - (1+xy)dz = 0$ **Ans.** $(x+y)z = c(x+z)$

Find the system of curves satisfying the differential equation

$$xdx + ydy + c\sqrt{\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)}dz = 0$$

which lie on the surface $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ **Ans.** $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ and $x^2 + y^2 + z^2 = c^2$.

UNIT 3

Partial Differential Equation

§ 5.1. Introduction :

Differential equations which involve two or more independent variables and partial derivatives of the dependent variable (or variables) with respect to these independent variables are called **partial differential equations.** Thus an equation relating partial derivatives is called a partial differential equation (p. d. equation).

As in the case of ordinary differential equations, the order of a p. d. equation is defined to be the order of the derivative of highest order occurring in the equation. If, for example, we take z to be the dependent variable and x, y and t to be independent variables then the equation

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial z}{\partial t} \qquad \ldots\ldots\ldots\ldots\ldots (1)$$

is a second order p. d. equation in two independent variables. Again, the equation

$$\left(\frac{\partial z}{\partial x}\right)^3 + \frac{\partial z}{\partial t} = 0 \qquad \ldots\ldots\ldots\ldots\ldots (2)$$

is a first order p. d. equation in two independent variables. Again, the equation

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} + \frac{\partial z}{\partial t} = 0 \qquad \ldots\ldots\ldots\ldots\ldots (3)$$

is a first order p. d. equation in three independent variables. A first order p. d. equation with z as dependent variable and x,y as independent variables can be written in symbolic form as

$$f(x,y,z,p,q,) = 0 \qquad \ldots\ldots\ldots\ldots\ldots (4)$$

where $$p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y} \qquad \ldots\ldots\ldots\ldots\ldots (5)$$

A common form of second order p.d. equation in two independent variables x,y is

$$Rr + Ss + Tt + Pp + Qq + F(x,y,z) = 0 \qquad \ldots\ldots\ldots\ldots (6)$$

where R,S,T,P,Q are functions of x and y and f is a prescribed function of x,y and z and

$$r = \frac{\partial^2 z}{\partial x^2}, \quad s = \frac{\partial^2 z}{\partial x \partial y}, \quad t = \frac{\partial^2 z}{\partial y^2} \qquad \ldots\ldots\ldots\ldots (7)$$

The general form of the equation (6) is

$$F(x,y,z,p,q,r,s,t) = 0 \qquad \ldots\ldots\ldots\ldots (8)$$

Again, p. d. equation may be divided into two broad classes, **linear equations** and **non-linear equations**. The degree of a p. d. equation is the exponent of the highest order. Such an equation is called **linear** if it is of the first degree in the dependent variable and its partial derivatives. That is, the powers and/or products of the dependent variable and its partial derivatives must be absent. An equation which is not linear is called a non-linear p.d. equation.

The equations (1), (3) and (6) are linear, but the equations (2), (4) and (8) are non-linear.

§ 5.2. Origin of partial differential equation :

Our interest here is to solve a first order p. d. equation of the type

$$P\frac{\partial z}{\partial x}+Q\frac{\partial z}{\partial y}=R \qquad \text{...............} (1)$$

or of the type $f(x,y,z,p,q)=0$ .... (2)

where P,Q,R are functions of x,y and $p=\frac{\partial z}{\partial x}$; $q=\frac{\partial z}{\partial y}$.

Obviously, the equation (1) is linear where as the equation (2) is non-linear in general. Before discussing the solution of equations of this type, we shall examine the interesting question of how partial differential equations arise. In what follows, it will be shown that such equations can be formed by elimination of arbitrary constants or arbitrary functions from an algebraic equation.

A. Elimination of arbitrary constants :

Consider a relation containing two independent variables x and y, the dependent variable z and two arbitrary constants a and b. Let this be

$$f(x,y,z,a,b)=0 \qquad \text{..............} (2)$$

By partial differentiation of equation (2) w.r.t. x and y in turn we get the equations

$$\frac{\partial f}{\partial x}\frac{\partial x}{\partial x}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial x}+\frac{\partial f}{\partial z}\frac{\partial z}{\partial x}=0 \quad \text{and} \quad \frac{\partial f}{\partial x}\frac{\partial x}{\partial y}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial y}+\frac{\partial f}{\partial z}\frac{\partial z}{\partial y}=0$$

Here $\frac{\partial y}{\partial x}=0=\frac{\partial x}{\partial y}$, since x and y are independent and $\frac{\partial x}{\partial x}=1=\frac{\partial y}{\partial y}$, $p=\frac{\partial z}{\partial x}, q=\frac{\partial z}{\partial y}$.

So the last two equations become

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial z}p=0 \qquad \text{...........}(3)$$

$$\frac{\partial f}{\partial y}+\frac{\partial f}{\partial z}q=0 \qquad \text{............} (4)$$

In general, the arbitrary constants a and b may be eliminated from the equations (2), (3) and (4). Thereby we get a partial differential equation of the first order

$$F(x,y,z,p,q) = 0 \qquad \ldots\ldots\ldots\ldots (5)$$

The equation (5) is often called the **eliminant** of the equation (2). It is in general, non-linear.

Worked Examples

Ex. 1. Obtain a partial differential equation of the first order by elimination of the constants a and b from the equation

$$axz + byz + abxy = 0$$

Solution : On differentiating the given equation partially w.r.t. x and y in turn, we obtain

$$az + ax\frac{\partial z}{\partial x} + by\frac{\partial z}{\partial x} + aby = 0 \quad \text{and} \quad ax\frac{\partial z}{\partial y} + bz + by\frac{\partial z}{\partial y} + abx = 0$$

or $\quad a(xp+z) + byp + aby = 0 \quad$ and $\quad axq + b(yq+z) + abx = 0$

Now, the given equation and the last two equations are linear homogeneous equations for the three quantities a,b and ab. For a non-trivial solution of these equations (that is all of a,b and ab cannot the zero), the determinant of their coefficient must vanish. This requires

$$\begin{vmatrix} xz & yz & xy \\ xp+z & yp & y \\ xq & yq+z & x \end{vmatrix} = 0$$

Expanding the determinant and simplifying we get $xp + yq = z$.

This is a first order linear p.d. equation and does not contain the constant a and b. This is therefore, the required eliminant of the given equation.

Ex. 2. Find two eliminants corresponding to the equation $z = ax + y/a$.

Solution : Differentiating the equation w.r.t. x, we get $p = \frac{\partial z}{\partial x} = a$.

Eliminating a from this and the given equation one obtains

$$z = px + y/p \qquad \text{or} \qquad xp^2 - zp + y = 0$$

which is an eleminant of the given equation. It is a first order non-linear p. d. equation. Next, differentiating the given equation w.r.t. y we get $\quad q = \frac{\partial z}{\partial y} = 1/a$

So the eliminant the partial differential equation

$$z = {}^{x}\!/_{q} + yq \qquad \text{or} \qquad x + y\,q^2 - zq = 0 \text{ which is the } 2^{nd} \text{ eliminant.}$$

Thus, we can obtain more than one eliminants i.e., p. d. equations from the same algebraic equation.

Note : The original algebraic equation from which we obtain eliminants as p.d. equation or equations is called the solutions of there p. d. equations. That means we can have the same solution for more than one p. d. equations.

Ex. 3. : Eliminate the constants a,b and c from the relation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \qquad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Solution : Differentiating (1) partially w.r.t. x and y we get

$$\frac{x}{a^2}+\frac{z}{c^2}p=0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

and $$\frac{y}{b^2}+\frac{z}{c^2}q=0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Differentiating (2) w.r.t x,

$$\frac{1}{a^2}+\frac{p^2}{c^2}+\frac{z}{c^2}p=0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

where $p=\frac{\partial z}{\partial x}, q=\frac{\partial z}{\partial y}, r=\frac{\partial^2 z}{\partial x^2}$

Now (4) × x − (2) gives $\qquad \frac{1}{c^2}=\left[p^2x+zxr-zp\right]=0$

or $\qquad pz = xp^2 + zxp$

which is an eliminant and is a non linear second order p. d. equation.

Alternatively, differentiating (3) partially w.r.t. y we get

$$\frac{1}{b^2}+\frac{z}{c^2}t+\frac{1}{c^2}q^2=0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

Now (5) × y − (3) gives $\qquad zq = yzt + yq^2$, where $t=\frac{\partial^2 z}{\partial y^2}$

This is another eliminant of the given equation.

B. Elimination of arbitrary functions :

Ex. 4. : Let u = u (x,y,z) and v = v (x,y,z) be two known functions of x,y and z and F be an arbitrary function of u and v such that

$$F(u,v) = 0 \qquad \text{..................} (1)$$

Find a p. d. equation by eliminanting F.

Solution : Treating z as dependent variable and x and y as independent varibales, we differentiate equation (1) partially with respect to x and y in turn to get

$$\frac{\partial F}{\partial u}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}p\right)+\frac{\partial F}{\partial v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}p\right)=0$$

$$\frac{\partial F}{\partial u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}q\right)+\frac{\partial F}{\partial v}\left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}q\right)=0$$

There are two linear homogeneous algebraic equations in $\frac{\partial F}{\partial u}$ and $\frac{\partial F}{\partial v}$. Since these quantities are not both to be zero, the determinant of the coefficients of the system must vanish. Hence we have,

$$\begin{vmatrix} \left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}p\right) & \left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}p\right) \\ \left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}q\right) & \left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}q\right) \end{vmatrix}=0$$

or $$\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}+q\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}\right)+p\left(\frac{\partial u}{\partial z}\frac{\partial v}{\partial y}\right)-\frac{\partial v}{\partial x}\frac{\partial u}{\partial y}-q\frac{\partial v}{\partial x}\frac{\partial u}{\partial z}-p\frac{\partial v}{\partial z}\frac{\partial u}{\partial y}=0 \text{ (on simplification)}$$

or $$\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial v}{\partial x}\frac{\partial u}{\partial y}\right)+p\left(\frac{\partial u}{\partial z}\frac{\partial v}{\partial y}-\frac{\partial v}{\partial z}\frac{\partial u}{\partial y}\right)+q\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}-\frac{\partial v}{\partial x}\frac{\partial u}{\partial z}\right)=0$$

or $$p\frac{\partial(u,v)}{\partial(y,z)}+q\frac{\partial(u,v)}{\partial(z,x)}=\frac{\partial(u,v)}{\partial(x,y)} \qquad \text{..................} (2)$$

This is linear p. d. equation of the first order and is the eliminant of F. Now putting $\frac{\partial(u,v)}{\partial(y,z)}=\lambda P$, $\frac{\partial(u,v)}{\partial(z,x)}=\lambda Q$, $\frac{\partial(u,v)}{\partial(x,y)}=\lambda R$, the equation (2) can be put as

$$Pp + Qz = R \qquad \text{..................} (3)$$

Ex. 5. : Find the eliminant of lowest order corresponding to the equation

$z = f(x + ay) + g(x - ay)$ where f and g are arbitrary functions and a is a constant.

Solution : Let u = x + ay, v = x – ay so that the given equation becomes

$$z = f(u) + g(v) \qquad \text{.................. (1)}$$

Differentiating it partially w.r.t. x and y we have

$$\frac{\partial z}{\partial x} = p = f'(u)\frac{\partial u}{\partial x} + g'(v)\frac{\partial v}{\partial x} = f'(u) + g'(v) \qquad \text{.......... (2)}$$

$$\frac{\partial z}{\partial y} = q = f'(u)\frac{\partial u}{\partial y} + g'(v)\frac{\partial v}{\partial y} = af'(u) - ag'(v) \qquad \text{.......... (3)}$$

The four quantities f, g, f ′, g′ cannot be eliminated from the three equations (1), (2) and (3).

So, we take the second partial derivatives as follow :

$$\frac{\partial^2 z}{\partial x^2} = r = f''(u) + g''(v) \qquad \text{.......... (4)}$$

$$\frac{\partial^2 z}{\partial y \partial x} = s = \frac{\partial q}{\partial x} = \frac{\partial p}{\partial y} = af''(u) - ag''(v) \qquad \text{.......... (5)}$$

and $$\frac{\partial^2 z}{\partial y^2} = t = \frac{\partial q}{\partial y} = a^2 f''(u) + a^2 g''(v) \qquad \text{.......... (6)}$$

Now eliminating f ″(u) and g″(v) from (4), and (6) we get

$$t = a^2 r \text{ or } \frac{\partial^2 z}{\partial x^2} - a^2 \frac{\partial^2 z}{\partial y^2} = 0$$ which is the required eliminant.

Ex. 6. : Find the eliminant of the equation $xy + yz + zx = f\left(\frac{z}{x+y}\right)$

where z is the dependent variable, x,y are independent variables and f is arbitrary.

Solution : On putting $u = \frac{z}{x+y}$, given equation becomes

$$xy + yz + zx = f\left(\frac{z}{x+y}\right) \qquad \text{.................. (1)}$$

Now $$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x} = \frac{-z}{(x+y)^2} \cdot 1 + p \cdot \frac{1}{(x+y)}$$

$$\frac{\partial u}{\partial y} = \frac{-z}{(x+y)^2} + q \cdot \frac{1}{(x+y)}$$

Differentiating the equation (1) partially w.r.t x and y in turn and using the above results we get

$$y+yp+z+xp=f'(u)\times\left[\frac{-z}{(x+y)^2}+\frac{p}{x+y}\right] \quad \ldots\ldots\ldots(2)$$

$$x+z+yq+qx=f'(u)\times\left[\frac{-z}{(x+y)^2}+\frac{q}{x+y}\right] \quad \ldots\ldots\ldots(3)$$

Eliminating f ′ (u) from (2) and (3), we get

$$[(y+z)+p(x+y)]\left[\frac{q}{x+y}-\frac{z}{(x+y)^2}\right]=[(x+z)+q(x+y)]\times\left[\frac{p}{x+y}-\frac{z}{(x+y)^2}\right]$$

or $[(y+z)+p(x+y)][q(x+y)-z]=[(x+z)+q(x+y)][p(x+y)-z]$

which on simplification, reduce to $(x+y)(x+2z)\,p-(x+y)(y+2z)\,q=(x-y)\,z$

This is the eliminant of the given equation and is a first order linear p.d. equation.

EXERCISE-1

1.Find all possible eliminants corresponding to the equation

$(x-a)^2+(y-b)^2+(z-c)^2=1$

Ans : $(p^2+q^2+1)(1+p^2)^2=r^2$, $(p^2+q^2+1)(1+q^2)^2=t^2$ and $(p^2+q^2+1)\,p^2q^2=s^2$

2. Formulate a partial differential equation by eliminating a,b and c from

$z=a(x+y)+b(x-y)+abt+c$ Ans : $\left(\frac{\partial z}{\partial x}\right)^2-\left(\frac{\partial z}{\partial y}\right)^2=4\frac{\partial z}{\partial t}$

3. Eliminate the arbitrary constants indicated in brackets from the following equations and from p. d. equations

(i) $z=Ae^{pt}\sin px$, (p and A), Ans : $\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial t^2}=0$

(ii) $Ae^{-2pt}\cos px$, (p and A), Ans : $\frac{\partial^2 z}{\partial x^2}=\frac{\partial z}{\partial t}$

(iii) $z=ax+by+a^2+b^2$, (a,b), Ans : $z=px+qy+p^2+q^2$

4. From a p. d. equation by eliminating the arbitrary function ϕ from

$\phi(x+y+z, x^2+y^2-z^2)=0$ Ans : $(y+z)p-(x+z)q=x-y$

5. Form p.d. equations by eliminating the arbitrary functions from the following equation :

(i) $x + y + z = f(x^2 + y^2 + z^2)$, Ans : $(y - z)p + (z - x)q = x - y$

(ii) $z = y^2 + 2 + f\left(\frac{1}{x} + \log y\right)$, Ans : $px^2 + qy = 2y^2$

(iii) $f(x^2 + y^2 + z^2, z^2 - 2xy) = 0$, Ans : $p - q = (y - x)/z$

(iv) $z = f(x + iy) + g(x - iy)$ where f,g are arbitrary functions Ans : $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$.

§ 5.3. Lagrange's linear p. d. equation and its geometrical interpretation :

The standard form of the linear partial differential equation of the first order and involving two independent variable z is

$$Pp + Qq = R \qquad \text{.................. (1)}$$

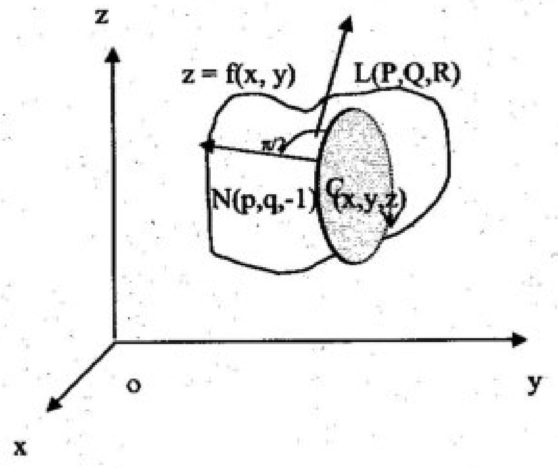
where P,Q,R are functions of x,y and z and $p = \frac{\partial z}{\partial x}, Q = \frac{\partial z}{\partial y}$.

This is often referred to as **Lagrange's linear equation .**

Geometrical meaning :

Let z = f (x,y) be a partial solution of equation (1). Consider a fixed point (x,y,z) on the integral surface z = f (x,y), we see that the direction cosines of the normal N to this surface at the point (x,y,z) are proportional to (p,q,-1).

Then the differential equation (1) means that the normal N is perpendicular to a line L through (x,y,z) and with direction ratios P,Q,R. In other words the direction (P,Q,R) is tangential to the integral surface z = f (x,y).



Now let the plane of N and L cut the surface in a curve C, having direction numbers dx, dy, dz. Since the curve C and the line L have the same direction, the two sets of direction numbers are proportional :

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R} \qquad \text{.................... (2)}$$

The simultaneous ordinary differential equations (2) are called the **subsidiary equations** for Lagrange's equation (1).

Lagrange's method of solution of equation (1) is based on this fact that there is a closed connection between the solutions of the subsidiary equation (2) and the integrals of the equation (1).

§ 5.4. The general integral :

Theorem : The general solution of the linear p. d. equation

$$Pp + Qq = R \qquad \text{.................... (1)}$$

is $\qquad \phi(u,v)=0 \qquad$ .... (2)

where ϕ is an arbitrary function and $u(x,y,z) = c_1$ and $v(x,y,z) = c_2$ form the solution of the equations

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R} \qquad \text{.................... (3)}$$

Proof : Given that $\quad u(x,y,z) = c_1 \quad$ .... (4)

satisfies the equations (3). Then taking total differential of (4) we get

$$u_x dx + u_y dy + u_z dz = 0 \qquad \text{.................... (5)}$$

Since (4) satisfies the equation (3), the equation (5) must be compatible with the equations $\quad \frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R} \quad$ .... (3)

So, we must have $\quad Pu_x + Qu_y + Ru_z = 0 \quad$ .... (6)

Similarly, from the other solution $v(x,y,z) = c_2$, we have

$$Pv_x + Qv_y + Rv_z = 0 \qquad \text{.................... (7)}$$

Solving the equation (6) and (7) for P,Q, and R we have

$$\frac{P}{u_y v_z - u_z v_y}=\frac{Q}{u_y v_x - u_x v_z}=\frac{R}{u_x v_y - u_y v_x}$$

or $$\frac{P}{\frac{\partial(u,v)}{\partial(y,z)}}=\frac{Q}{\frac{\partial(u,v)}{\partial(z,x)}}=\frac{R}{\frac{\partial(u,v)}{\partial(x,y)}} \quad ....(8)$$

Again, it can be shown that (section 5.2.B. Ex. 4.) the relation

ϕ (u,v) = 0 leads to the p. d. equation

$$p\frac{\partial(u,v)}{\partial(y,z)}+q\frac{\partial(u,v)}{\partial(z,x)}=\frac{\partial(u,v)}{\partial(x,y)} \quad ....(9)$$

Substituting from equations (8) and (9) we get $Pp+Qq=R$ which is equation (1).

Thus we may conclude that the relation (2) is a solution of the p. d. equation (1) where $u = c_1$ and $v = c_2$ are two integrals of the subsidiary equation (3). The solution $\phi(u,v) = 0$, ϕ being arbitrary is called the general integral of equation (1).

Note : **The two integrals** $u = c_1$ and $v = c_2$ are also solutions of the p. d. equation (1).

Proof : Let u and v involve z explicitly. Also let $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}=d\lambda$

so that $dx = Pd\lambda,\ dy = Qd\lambda,\ dz = Rd\lambda$ .... (10)

Taking total differential of u (x,y,z) = c_1, we get $u_x\,dx + u_y\,dy + u_z\,dz = 0$

or using (10) and canceling $d\lambda$, $Pu_x + Qu_y + Ru_z = 0$

or $$-P\frac{u_x}{u_z}-Q\frac{u_y}{u_z}=R \quad ....(11)$$

Again from u (x,y,z) = c_1, we have $p=\frac{\partial z}{\partial x}=-\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial z}}=-\frac{u_x}{u_z},\ q=\frac{u_y}{u_z}$

Hence (11) may be written as $Pp + Qq = R$ .... (1)

Thus the integral u (x,yz) = c_1 of the subsidiary equation (3) satisfies the Lagrange's equation (1). Similarly, the integral v (x,y,z) = c_2 of (3) is also a solution of equation (1).

Worked Examples

Ex. 1. : Find the general solution of the differential equation

$$x^2\frac{\partial z}{\partial x}+y^2\frac{\partial z}{\partial y}=(x+y)z \quad ....(1)$$

Solution : The given equation is a linear first order p.d. equation. Its subsidiary equations are

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{(x+y)z} \qquad \text{.....................} (2)$$

The first pair of the ratios in (2) gives $\frac{dx}{x^2} = \frac{dy}{y^2}$

Integrating, $\frac{1}{x} - \frac{1}{y} = \text{constant} = c_1$ .... (3)

Again, we have from (2), $\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{(x+y)z} = \frac{dx - dy}{(x^2 - y^2)}$

$$\therefore \quad \frac{dz}{z} = \frac{d(x-y)}{(x-y)} \quad \text{or} \quad \frac{dz}{z} = \frac{d(x-y)}{(x-y)}$$

Integrating, $\frac{x-y}{z} = c_2$ .... (4)

From (3) and (4) we get $\frac{xy}{z} = \frac{c_2}{c_1} = c_3$ .... (5)

The integral curves of the equations (2) are given by the equations (4) and (5).

The general solution of the given differential equation (1) is given by

$F\left(\frac{xy}{z}, \frac{x-y}{z}\right) = 0$ .... (6) where the function F is arbitrary.

Ex. 2. Find the general integral of the Lagrange equation

$$x(z^2 - y^2)p + y(x^2 - z^2)q = z(y^2 - x^2) \qquad \text{............} (1)$$

Solution : The subsidiary equations are

$$\frac{dx}{x(z^2 - y^2)} = \frac{dy}{y(x^2 - z^2)} = \frac{dz}{z(y^2 - x^2)} \qquad \text{............} (2)$$

Each ratio in (2) is

$$= \frac{xdx + ydy + zdz}{x^2(z^2 - y^2) + y^2(x^2 - z^2) + z^2(y^2 - x^2)} = \frac{xdx + ydy + zdz}{0}$$

$\therefore$ xdx + ydy + zdz = 0

Integrating,

$$x^2 + y^2 + z^2 = a^2 \qquad \text{.............} (3)$$

Next, each ratio of (2) is

$$= \frac{yzdx + zxdy + xydz}{xyz[(z^2 - y^2) + (x^2 - z^2) + (y^2 - x^2)]} = \frac{yzdx + zxdy + xydz}{0}$$

$\therefore$ $yzdx + zxdy + xydz = 0$ or $d(xyz) = 0$

Integrating, $xyz = c$ .... (4)

$\therefore$ the general integral of the given equation is $F(x^2 + y^2 + z^2, xyz) = 0$ where F is arbitrary.

Ex. 3. Find the general integral of the p. d. equation

$$z - xp - yq = a\sqrt{x^2 + y^2 + z^2} \qquad \text{................ (1)}$$

Solution : The subsidiary equation of (1) are

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z - a\sqrt{x^2 + y^2 + z^2}} \qquad \text{................ (2)}$$

The first two ratios give $\frac{dx}{x} = \frac{dy}{y}$. Integrating $y = c_1 x$ ....(3)

Next, each member of (2) is $= \dfrac{xdx + ydy + zdz}{x^2 + y^2 + z^2 - az\sqrt{(x^2 + y^2 + z^2)}}$, equating the last ratio of (2) with this, we get $\dfrac{dz}{z - a\sqrt{x^2 + y + z^2}} = \dfrac{xdx + ydy + zdz}{(x^2 + y^2 + z^2) - az\sqrt{(x^2 + y^2 + z^2)}}$

Put $x^2 + y^2 + z^2 = t^2$ so that the last equation reduces to

$$\frac{dz}{z - at} = \frac{\frac{1}{2}dt^2}{t^2 - azt} = \frac{tdt}{t(t - az)} = \frac{dt}{t - az}$$

$$\therefore \quad \frac{dx}{x} = \frac{dy}{y} = \frac{dz + dt}{(z + t) - a(t + z)} = \frac{d(z + t)}{(1 - a)(z + t)}$$

$$\text{or} \quad (1 - a)\frac{dx}{x} = \frac{d(z + t)}{(z + t)}$$

Integrating, $(1 - a) \log x = \log (z + t) + \log c_2$

or $\quad x^{1-a} = c_2 (z + t) = c_2 \left(z + \sqrt{x^2 + y^2 + z^2}\right)$

$$\text{or} \quad \frac{x^{1-a}}{z + \sqrt{(x^2 + y^2 + z^2)}} = c_2 \qquad \text{.................. (4)}$$

From (3) and (4) we get the general integral of equation (1) as

$$F\left(y/x , \frac{x^{1-a}}{z+\sqrt{x^2+y^2+z^2}} \right) = 0$$

§ 5.5. Integral surfaces passing through a given curve

The general solution of a linear p. d. equation can be used to determine the integral surface which passes through a given curve.

Suppose we are given the p. d. equation

$$Pp + Qq = R \qquad \text{..................} (1)$$

Its subsidiary equation are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \qquad \text{..................} (2)$$

Let the two solutions of equation (2) be

$$u(x,y,z) = c_1,\ v(x,y,z) = c_2 \qquad \text{..............} (3)$$

Then we know that the general integral of the p. d. equation (1) is

$$F(u,v) = 0 \qquad \text{..................} (4)$$

arising from the relation $F(c_1, c_2) = 0$ .... (5)

between the constants c_1 and c_2. The function F is arbitrary.

Now, our problem is to determine the function F in special circumstances.

Suppose, we wish to find the integral surface which passes through the curve C whose parametric equations are

$$x = x(t),\ y = y(t),\ z = z(t)$$

where t is a parameter.

Then the particular solution (3) must be such that

$$u\{x(t), y(t), z(t)\} = c_1,\ v = \{x(t), y(t), z(t)\} = c_2$$

We therefore have two equations from which we may eliminant the single variable t to obtain a relation of the type (5). Then our required solution is given by the equation (4). This equation will not involve any arbitrary constant and so will represent a definite surface with 3-dimensional co-ordinate spaces.

Ex. 4. Find the integral surface of linear p. d. equation

$$x(y^2+z)p - y(x^2+z)q = (x^2-y^2)z$$

which contains the straight line $x + y = 0$, $z = 1$.

Solution : The auxiliary equation of the given equation are

$$\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{(x^2-y^2)z}$$

each ratio is $= \dfrac{yzdx + zxdy + xydy}{xyz\left[y^2+z-x^2-z+x^2-y^2\right]} = \dfrac{d(xyz)}{xyz*0}$ $\quad \therefore\ d(xyz) = 0$

Integrating $x\,y\,z = c_1$ .... (1)

Again each ratio is $= \dfrac{xdx + ydy - dz}{x^2(y^2+z) - y^2(x^2+z) - (x^2-y^2)z} = \dfrac{\frac{1}{2}d(x^2+y^2-2z)}{0}$

$\therefore d(x^2+y^2-2z) = 0$

Integrating $x^2 + y^2 - 2z = c_2$ .... (2)

Thus the solution of the subsidiary equations are given by

$xyz = c_1,\ x^2 + y^2 - 2z = c_2$ .... (3)

Again for the given straight line, we may choose the parametric equation as

$x = t,\ y = -t,\ z = 1$

Substituting these in equation (3) we get $\quad -t^2 = c_1,\ 2t^2 - 2 = c_2$

Eliminating t from these, we find $\quad -2c_1 - 2 = c_2 \quad$ or $\quad 2c_1 + c_2 + 2 = 0$

Then making use of the equations (3) we have

$$2xyz + (x^2 + y^2 - 2z) + 2 = 0$$

or $\quad x^2 + y^2 + 2xyz - 2z + 2 = 0 \quad$ which gives the desired integral surface.

Ex. 5. Find the integral surface of the equation

$(x - y)y^2p + (y - x)x^2q = (x^2 + y^2)z$ .... (1)

through the curve $xz = a^3$, $y = 0$ .... (2)

Solution : The auxiliary equations are

$$\frac{dx}{(x-y)y^2} = \frac{dy}{(y-x)x^2} = \frac{dz}{(x^2+y^2)z} \quad \text{.............} (3)$$

The first two members gives $\frac{x^2dx}{x-y} = \frac{y^2dy}{-(x-y)}$

or $\quad x^2dx + y^2\,dy = 0$

Integrating $x^3 + y^3 = c_1$ .... (4)

Again each ratio in (3) is $= \frac{dx - dy}{(x-y)(y^2+x^2)}$

Equating this with the last ratio of (3),

$$\frac{dz}{(x^2+y^2)z} = \frac{d(x-y)}{(x-y)(x^2+y^2)} \quad \text{or} \quad \frac{dz}{z} = \frac{d(x-y)}{(x-y)}$$

Integrating, $z = c_2\,(x - y)$ .... (5)

Thus the solution of the equation (3) is given by

$$x^3 + y^3 = c_1,\ z = c_2\,(x - y) \quad \text{...............} (6)$$

For the given curve (2), we take the parametric equation

$$x = t, \qquad z = \frac{a^3}{t}, \qquad y = 0$$

Substituting these in (6) we have

$t^3 + 0 = c_1$ i.e. $t^3 = c_1$ and

$\frac{a^3}{t} = c_2(t-0)$ on $c_2 = \frac{a^3}{t^2}$ i.e. $t^2 = \frac{a^3}{c_2}$

Eliminating t from these two equations we have

$$c_1^2 = t^6 = \left(\frac{a^3}{c_2}\right)^2 = \frac{a^9}{c_2^3}$$

Then substituting from (6), we have

$$(x^3+y^3)^2 = a^9\left(\frac{x-y}{z}\right)^3 \quad \text{or} \quad z^3(x^3+y^3)^2 = a^9(x-y)^3$$

This equation represents the required integral surface.

Ex. 6. Find the general solution of the equation

$$2x(y+z^2)p + y(2y+z^2)q = z^3 \qquad \ldots\ldots\ldots\ldots\ldots (1)$$

and deduce that

$$yz(z^2 + yz - 2y) = x^2 \qquad \ldots\ldots\ldots\ldots\ldots (2)$$

is a solution.

Solution: The subsidiary equations are

$$\frac{dx}{2x(y+z^2)} = \frac{dy}{y(2y+z^2)} = \frac{dz}{z^3} \qquad \ldots\ldots\ldots\ldots\ldots (3)$$

Two solutions of the equations (3) can be obtained as

$$\frac{x}{yz} = c_1 \qquad \ldots\ldots\ldots\ldots\ldots (4)$$

and $$\frac{z^2 - 2y}{yz} = c_2 \qquad \ldots\ldots\ldots\ldots\ldots (5)$$

Hence, the general solution of equation (1) is

$$F\left(\frac{x}{yz}, \frac{z^2-2y}{yz}\right) = 0 \qquad \ldots\ldots\ldots\ldots\ldots (6)$$

Second part : The equation (2) can be written as

$$yz(z^2 + yz - 2y) = x^2$$

or $$\frac{1}{yz}\{(z^2 - 2y) + yz\} = \frac{x^2}{(yz)^2} = \left(\frac{x}{yz}\right)^2$$

or $$\frac{z^2 - 2y}{yz} + 1 = \left(\frac{x}{yz}\right)^2$$

or $$c_2 + 1 = c_1^2 \qquad \ldots\ldots\ldots \text{ using (4) and (5)}$$

or $$c_1^2 - c_2 - 1 = 0$$

which is of the form $F_1(c_1, c_2) = 0$

Thus the given equation (2) is a particular solution of the Lagrange equation (1).

§ 5.6. Linear equations with n-independent variables

The first order linear p. d. equation in the n-independent variables

$x_1, x_2, \ldots\ldots, x_n$ and the dependent variable z is of the form

$$P_1p_1 + P_2p_2 + \cdots + P_np_n = R \qquad \ldots\ldots\ldots\ldots\ldots (1)$$

where $p_j = \dfrac{\partial z}{\partial x_j}, j = 1,2,\ldots\ldots,n$ and P_i's and R are functions of the x's and z.

When n = 2, the equation (1) reduces to Lagrange's equation.

The subsidiary equations of the equation (1) are

$$\frac{dx_1}{P_1} = \frac{dx_2}{P_2} = \cdots = \frac{dx_n}{P_n} = \frac{dz}{R} \qquad \ldots\ldots\ldots\ldots\ldots (2)$$

These n-independent ordinary differential equations in (2) will generally have n-independent integrals

$u_j(x_1, x_2, \ldots\ldots, x_n, z) = a_j, j = 1,2,\ldots\ldots,n.$ where a's are arbitrary constants. When n > 2, it may be shown by the some method employed for the Lagrange equation that

$$\phi(u_1, u_2, \ldots\ldots u_n) = 0 \qquad \ldots\ldots\ldots\ldots\ldots (3)$$

where ϕ is an arbitrary function of its arguments is a solution of equation (1). The relation (3) is called the general integral of the linear partial differential equation (1). A relation involving n-arbitrary constants and satisfying equation (1) is called a complete integral integral of equation (1).

When n > 2, equation (1) and its solutions do not have geometric interpretation as do the Lagrange equation and its solution. However, the equation (1) is said to be (n + 1) dimensional.

Worked Examples

Ex. 7. Solve $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} + t\dfrac{\partial z}{\partial t} = az + \dfrac{xy}{t} \qquad \ldots\ldots\ldots\ldots\ldots (1)$

Solution : The auxiliary equations of equation (1) are

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dt}{t} = \frac{dz}{az + xy/t}$$

Now, $\dfrac{dx}{x} = \dfrac{dy}{y} \Rightarrow y = c_1x \qquad \ldots\ldots\ldots\ldots\ldots (2)$

$$\frac{dx}{x}=\frac{dt}{t}\Rightarrow t=c_2x \qquad \text{.................}(3)$$

Then from $\quad \dfrac{dx}{x}=\dfrac{dz}{az+{}^{xy}\!/_{t}}=\dfrac{dz}{az+\dfrac{c_1x^2}{c_2x}}=\dfrac{dz}{az+\dfrac{c_1}{c_2}x}$

or $\quad \dfrac{dz}{dx}=\dfrac{az+\dfrac{c_1}{c_2}x}{x}=a\dfrac{z}{x}+\dfrac{c_1}{c_2}$

or $\quad \dfrac{dz}{dx}-a\dfrac{z}{x}=\dfrac{c_1}{c_2} \qquad \text{................}(4)$

This is a linear first order ordinary differential equation.

Its I.F. $=e^{\int -{}^{a}\!/_{x}du}=e^{-a\log x}=x^{-a}$

Hence, $zx^{-a}=c_3+\int\dfrac{c_1}{c_2}x^{-a}dx=c_3+\dfrac{c_1}{c_2}\times\dfrac{x^{1-a}}{1-a}$

or, $\quad z=x^{a}\left[c_3+\dfrac{y}{t}\dfrac{x^{1-a}}{1-a}\right] \qquad \text{......}\because \dfrac{c_1}{c_2}=\dfrac{y}{t}$

i.e., $\quad \dfrac{z}{x^a}-\dfrac{y}{t}\dfrac{x^{1-a}}{1-a}=c_3 \qquad \text{.................}(5)$

Thus (2), (3) and (5) are the integral of the auxiliary equation. Hence the general integral of the given p. d. equation is

$$\phi\left({}^{y}\!/_{x},{}^{t}\!/_{x},\frac{z}{x^a}-\frac{y}{t}\times\frac{x^{1-a}}{1-a}\right)=0 \quad \text{where } \phi \text{ is arbitrary.}$$

Ex. 8. Solve $z(x_2x_3p_1+x_1x_3p_2+x_1x_2p_3)=x_1x_2x_3$

Solution : The given equation is

$$zx_2x_3\frac{\partial z}{\partial x_1}+zx_1x_3\frac{\partial z}{\partial x_2}+zx_1x_2\frac{\partial z}{\partial x_3}=x_1x_2x_3 \qquad \text{............}(1)$$

The auxiliary equations are

$$\frac{dx_1}{zx_2x_3}=\frac{dx_2}{zx_1x_3}=\frac{dx_3}{zx_1x_2}=\frac{dz}{x_1x_2x_3} \qquad \text{............}(2)$$

From the first two ratios of (2),

$$\frac{dx_1}{zx_2x_3} = \frac{dx_2}{zx_1x_3} \text{ or } x_1dx_1 = x_2dx_2$$

$\therefore$ Integrating, $x^2_1 - x^2_2 = c_1$ .... (3)

Similarly, from the 2nd and 3rd ratios of (2),

$x^2_2 - x^2_3 = c_2$ .... (4)

Then from the last two ratios of (2),

$x^2_3 - z^2 = c_3$ .... (5)

Now (3),(4) and (5) are the three independent integrals of the subsidiary equations.

Hence, the general solution of the given equation is

$$\phi(x_1^2 - x_2^2, x_2^2 - x_3^2, x_3^2 - z^2) = 0$$

§ 5.7. Homogeneous equations lacking the dependent variable

A homogeneous linear p. d. equation of the first order with coefficients free from the dependent variable F is an equation of the form

$$Q_1\frac{\partial F}{\partial x_1} + Q_2\frac{\partial F}{\partial x_2} + \cdots + Q_n\frac{\partial F}{\partial x_n} = 0 \qquad \text{.............. (1)}$$

where $Q_1, Q_2, \ldots\ldots, Q_n$ are functions of the n-independent variables $x_1, x_2, \ldots, x_n$, but do not involve F.

The subsidiary equations of (1) are

$$\frac{dx_1}{Q_1} = \frac{dx_2}{Q_2} = \cdots = \frac{dx_n}{Q_n}, dF = 0 \qquad \text{.............. (2)}$$

If $u_j\ (x_1, x_2, \ldots\ldots, x_n) = a_j$, $j = 1, 2, \ldots\ldots, n-1$ involving only the x's are $n-1$ independent integrals of (2), then the general integral of (1) is given by

$$F = \psi\,(u_1, u_2, \ldots\ldots, u_{n-1}) \qquad \text{.............. (3)}$$

where ψ is arbitrary.

Ex. 9. If u is a function of x,y and z which satisfies the p. d. equation

$$(y-z)\frac{\partial u}{\partial x} + (z-x)\frac{\partial u}{\partial y} + (x-y)\frac{\partial u}{\partial z} = 0$$

so that u contains x,y and z only in combination of $x + y + z$ and $x^2 + y^2 + z^2$.

Solution : The given p. d. equation has the auxiliary equations

$$\frac{dx}{y-z}=\frac{dy}{z-x}=\frac{dz}{x-y}=\frac{du}{0}$$

From these we obtain

$$du = 0$$

$$dx + dy + dz = 0$$

and $$xdx + ydy + zdz = 0$$

The integrals are

$$u = c_1, \qquad x + y + z = c_2, \qquad x^2 + y^2 + z^2 = c_3$$

Hence the general solution is of the form

$u = f(x + y + z, x^2 + y^2 + z^2)$ as required to be shown.

Exercise – 2

1.Find the general integrals of the following Lagrange equations :

(i) $(y^2 - z^2)p + (z^2 - x^2)q = x^2 - y^2$ Ans : $\phi(x + y + z, x^3 + y^3 + z^3) = 0$

(ii) $(z^2 - 2yz - y^2)p + (xy + xz)q = xy - xz$ Ans: $\phi(x^2 + y^2 + z^2, y^2 - 2yz - z^2) = 0$

(iii) $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$ Ans : $\phi\left(\frac{x-y}{y-z}, \frac{y-z}{z-x}\right) = 0$

(iv) $(mz - ny)p + (nx - lz)q = ly - mx$ Ans : $\phi(lx + my + nz, x^2 + y^2 + z^2) = 0$

(v) $(x^3 + 3xy^2)p + (y^3 + 3x^2y)q = 2(x^2 + y^2)z$ Ans : $\phi\left\{\frac{1}{(x-y)^2} - \frac{1}{(x+y)^2}, \frac{xy}{z^2}\right\} = 0$

2. Find the general integral of the p. d. equation

$$(2xy - 1)p + (z - 2x^2)q = 2(x - yz)$$

and also the particular integral which passes through the line $x = 1, y = 0$.

Ans : $F(y + zx, x^2 - 2xyz - x^2z^2 + z) = 0$

$$x^2 + y^2 - zx - y + z - 1 = 0$$

3. Find the equation of the integral surface of the differential equation

$$2y(z - 3)p + (2x - z)q = y(2x - 3)$$

which passes through the circle $z = 0, x^2 + y^2 = 2x$.

Ans : $x^2 + y^2 - 2x = z^2 - 4z$.

4. Find the general integral of the p. d. equation $(x - y)p + (y - x - z)q = z$ and the particular solution through the circle $z = 1, x^2 + y^2 = 1$.

Ans : $\phi\left(x+y+z, \frac{y-x-z}{z^2}\right) = 0$

$$z^4(x+y+z)^2 - 2z^4(x+y+z) + (y-x-z)^2 + 2z^2(y-x-z) = 0$$

5. Solve :

(i) $(t+y+z)\frac{\partial t}{\partial x} + (t+z+x)\frac{\partial t}{\partial y} + (t+x+y)\frac{\partial t}{\partial z} = x+y+z$

Ans : $\phi\left[\frac{y-z}{x-y}, \frac{t-z}{y-z}, (x+y+z+t)^{1/3}(t-z)\right] = 0$

(ii) $x_2x_3p_1 + x_3x_1p_2 + x_1x_2p_3 + x_1x_2x_3 = 0$

(here z is the dependent variable)

Ans : $F(x_1^2 + 2z, x_1^2 - x_2^2, x_2^2 - x_3^2) = 0$

(iii) $p_1 + p_2 + p_3\{1 - \sqrt{(z - x_1 - x_2 - x_3)}\} = 3$

Ans : $\phi\{z - 3x_1, z - 3x_2, z + 6\sqrt{(z - x_1 - x_2 - x_3)}\} = 0$

(iv) $(x_3 - x_2)p_1 + x_2p_2 - x_3p_3 = x_2(x_1 + x_3) - x_2^2$

Ans : $F(z - x_1x_2, x_1 + x_2 + x_3, x_2x_3) = 0$

(v) $x_3p_1 + x_2p_2 + x_1p_3 = 0$

Ans : $\phi\left\{(x_1 + x_3)/x_2, x_1^2 - x_3^2\right\} = 0$

(vi) $x_2x_3p_1 + x_1x_3p_2 + x_1x_2p_3 = 0$

Ans : $\phi(x_1^2 - x_2^2, x_2^2 - x_3^2, z) = 0$

(vii) $x_1p_1 + 2x_2p_2 + 3x_3p_3 + 4x_4p_4 = 0$

Ans : $\phi\left(z, x_1^2/x_2, x_1^3/x_3, x_1^4/x_4\right) = 0$

UNIT 4

Non-linear partial differential equations of the first order; Charpit's method of solution.

§ 6.1. introduction

Let us consider the non-linear p. d. equation in two independent variables x,y and the dependent variable z, denoted by

$$F(x, y, z, p, q) = 0 \qquad \text{.............} (1)$$

where $p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}$ and F is not a linear function in p and q.

We know that the partial differential equation of the two parameter system of surfaces

$$f(x,y,z,a,b) = 0 \qquad \text{.............} (2)$$

is of the form (1). It can also be shown that the converse is also true (Charpit's method to be discussed hereafter), that is, that any partial differential equation of the type (1) has solutions of the type (2). Such a solution containing two arbitrary constants (here a and b) is called a **complete integral** of the partial differential equation (1).

In general, the arbitrary constants a and b in equation (2) will not occur linearly. So, the two parameter family of surface (2) may posses an envelope. This envelope is obtained from

$$f = 0, \frac{\partial f}{\partial a} = 0 \text{ and } \frac{\partial f}{\partial b} = 0 \qquad \text{...............} (3)$$

by elimination of a and b amongst them.

The envelope will touch at each of its points, a member of the system (2). They possess, therefore, the same set of values (x,y,z,p,q) as the particular surface, so that it must also be a solution of the differential equation (1).

Thus the envelope of the two parameter system (2) when it exists, is also a solution of the equation (1) and it is called the **singular integral** of the p. d. equation (1).

§ 6. 2. Charpit's method for solution of a non-linear first order p.d. equation with two independent variables.

To solve the non-linear first order p. d. equation $F(x,y,z,p,q) = 0$ .... (1)

Charpit's method of solving this equation is based on finding a second p. d. equation of the first order $\quad G(x,y,z,a,b) = 0 \quad$ .... (2)

such that equations (1) and (2) can be solved for p and q in terms of x,y and z and such that these resulting expressions, when inserted in $\quad p(x,y,z)dx + q(x,y,z)dy - dz = 0$ ..... (3)

makes (3) integrable.

In order that (1) and (2) be solvable for p and q, these relations have to be independent and so their Jacobian

$$J = \frac{\partial(F,G)}{\partial(p,q)} = \begin{vmatrix} \frac{\partial F}{\partial p} & \frac{\partial G}{\partial p} \\ \frac{\partial F}{\partial q} & \frac{\partial G}{\partial q} \end{vmatrix} = F_p G_q - F_q G_p \qquad \ldots\ldots\ldots\ldots (4) \text{ cannot vanish identically.}$$

Again, the necessary and sufficient condition for equation (3) be integrable is

$$\begin{vmatrix} p & q & -1 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ p & q & -1 \end{vmatrix} = 0$$

or $$-p\left(\frac{\partial q}{\partial z}\right) + q\left(\frac{\partial p}{\partial z}\right) - 1\left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right) = 0$$

or $$p\frac{\partial q}{\partial z} - q\frac{\partial p}{\partial z} - \frac{\partial p}{\partial y} + \frac{\partial q}{\partial x} = 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

Now, differentiating (1) and (2) partially w.r.t. x (holding x and z fixed) we get,

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial F}{\partial q}\frac{\partial q}{\partial x} = 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots (6)$$

and $$\frac{\partial G}{\partial x} + \frac{\partial G}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial G}{\partial q}\frac{\partial q}{\partial x} = 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots (7)$$

Solving (6) and (7) for $\frac{\partial q}{\partial x}$ we get $$\frac{\partial q}{\partial x} = \frac{\frac{\partial F}{\partial x}\frac{\partial G}{\partial p} - \frac{\partial G}{\partial x}\frac{\partial F}{\partial p}}{-\frac{\partial F}{\partial q}\frac{\partial G}{\partial p} + \frac{\partial G}{\partial q}\frac{\partial F}{\partial p}} = \frac{F_x G_p - F_p G_x}{J} \quad \ldots\ldots (8)$$

Similarly, differentiation of (1) and (2) partially w.r.t. y gives,

$$F_y + F_p\frac{\partial p}{\partial y} + F_q\frac{\partial q}{\partial y} = 0 \qquad \text{and} \qquad G_y + G_p\frac{\partial p}{\partial y} + G_q\frac{\partial q}{\partial y} = 0$$

Solving these two equations for $\frac{\partial p}{\partial y}$, one obtains $$\frac{\partial p}{\partial y} = \frac{F_q G_y - F_y G_q}{J} \quad \ldots\ldots\ldots\ldots (9)$$

Next, differentiating (1) and (2) partially w.r.t. z, we get

$$F_z + F_p\frac{\partial p}{\partial z} + F_q\frac{\partial q}{\partial z} = 0 \qquad \text{and} \qquad G_z + G_p\frac{\partial p}{\partial z} + G_q\frac{\partial q}{\partial z} = 0$$

These can be solved for $\frac{\partial p}{\partial z}$ and $\frac{\partial q}{\partial z}$ to yield

$$\frac{\partial p}{\partial z} = (F_q G_z - F_z G_q)\Big/J \text{ , and } \frac{\partial q}{\partial z} = \frac{F_z G_p - F_p G_z}{J} \quad \text{...... (10)}$$

Now inserting the expressions for $\frac{\partial q}{\partial x}, \frac{\partial p}{\partial y}, \frac{\partial p}{\partial z}$ and $\frac{\partial q}{\partial z}$ from (8) to (10) in (5), and multiplying by J, we get,

$$p(F_z G_p - F_p G_z) - q(F_q G_z - F_z G_q) - (F_q G_y - F_y G_q) + (F_x G_p - F_p G_x) = 0$$

or $$(F_x + pF_z)\frac{\partial G}{\partial p} + (F_y + qF_z)\frac{\partial G}{\partial q} - (pF_p + qF_q)\frac{\partial G}{\partial z} - F_p \frac{\partial G}{\partial x} - F_q \frac{\partial G}{\partial y} = 0 \quad \text{.........(11)}$$

This is a six-dimensional linear homogeneous p. d. equation for G as a function of x, y, z, p and q. Its subsidiary equation are

$$\frac{dp}{F_x + pF_z} = \frac{dq}{F_y + qF_z} = \frac{dz}{-pF_p - qF_q} = \frac{dx}{-F_p} = \frac{dy}{-F_q}, dG = 0 \quad \text{.......(12)}$$

These equations are known as Charpit's equation. Finally, since any integral of these equations (12) is a solution of the equation (11), we try to find one integral of (12) containing p or q or both. Let such an integral be $u(x,y,z,p,q) = \alpha$ .... (13) where α is a constant. This gives us the relation (2), that is $G = u - \alpha = 0$.

We then solve (1) and (3) simultaneously for p and q as functions of x,y,z and α. Substitute these in $pdx + qdy - dz = 0$ ....(13)

and integrate it to get $f(x,y,z,\alpha,\beta) = 0$ .... (14)

where β is a second arbitrary constant.

The solution (14) containing two arbitrary constants will be a complete integral of the equation (1). A singular integral, if it exists, and a particular case of the general integral may then be found.

§ 6. 3. Worked Examples

Ex. 1. Find a complete integral of the equation

$$p^2 x + q^2 y = z \quad \text{..................(1)}$$

Solution : The equation is non-linear 1st order p. d. equation & $F(x,y,z,p,q) = p^2 x + q^2 y - z$.

So that,

$$F_x = \frac{\partial F}{\partial x} = p^2, \quad F_y = \frac{\partial F}{\partial y} = q^2, \quad F_z = \frac{\partial F}{\partial z} = -1, \quad F_p = \frac{\partial F}{\partial p} = 2px, \quad F_q = \frac{\partial F}{\partial q} = 2qy$$

The auxiliary equations of (1) are

$$\frac{dp}{F_x + pF_z} = \frac{dq}{F_y + qF_z} = \frac{dz}{-pF_p - qF_q} = \frac{dx}{-F_p} = \frac{dy}{-F_q} \text{ and } dG = 0$$

or $$\frac{dp}{p^2 - p} = \frac{dq}{q^2 - q} = \frac{dz}{-p.2px - q.2qy} = \frac{dx}{-2px} = \frac{dy}{-2qy} \text{ and } dG = 0 \quad \ldots\ldots\ldots\ldots (2)$$

Each ratio in (2) is $= \dfrac{p^2dx + 2pxdp}{-2p^3x + 2p^3x - 2p^3x} = \dfrac{p^2dx + 2pxdp}{-2p^2x}$

Also each ratio in (2) is $= \dfrac{q^2dy + 2qydq}{-2q^3y + 2q^3y - 2q^2y} = \dfrac{q^2dy + 2qydy}{-2q^2y}$

Equating the last two ratios we get

$$\frac{p^2dx + 2pxdp}{-2p^2x} = \frac{q^2dy + 2qydy}{-2q^2y} \qquad \text{or} \qquad \frac{d(p^2x)}{p^2x} = \frac{d(q^2y)}{q^2y}$$

Integrating, $\log(p^2x) = \log(q^2y) + \log a$, where a is an arbitrary constant of integration. $\therefore p^2x = a\,q^2y$ .... (3)

Now we have to solve equations (1) and (3) for p and q. (1) × a + (3) gives

$$ap^2x + aq^2y + p^2x = az + qq^2y$$

$$\text{or} \quad p^2(1+a)x = az \text{ or } p^2 = \frac{az}{(1+a)x} \qquad \therefore p = \pm\left\{\frac{az}{(1+a)x}\right\}^{1/2}$$

Then (3) gives $\quad aq^2y = p^2x = \dfrac{az}{1+a}$ or $q^2 = \dfrac{z}{(1+a)y} \qquad \therefore q = \pm\left\{\dfrac{z}{(1+a)y}\right\}^{1/2}$

Substituting these in the equation

$dz = pdx + qdy$, we obtain

$$dz = \left\{\frac{az}{(1+a)x}\right\}^{1/2} dx + \left\{\frac{z}{(1+a)y}\right\}^{1/2} dy \text{ (taking the sign for p and q only)}$$

or $$(1+a)^{1/2}\frac{dz}{z^{1/2}} = a^{1/2}\frac{dx}{x^{1/2}} + \frac{dy}{y^{1/2}}$$

Integrating, $$(1+a)^{1/2}\frac{z^{1/2}}{1/2} = a^{1/2}\frac{x^{1/2}}{1/2} + \frac{y^{1/2}}{1/2} + \text{constant}$$

or $\quad \{(1+a)z\}^{1/2} = (ax)^{1/2} + y^{1/2} + b$ ....(4)

where b is another arbitrary constant of integration. Equation (4) is the complete integral of the equation (1).

Ex. 2. Find the complete integral of

$$2(z + xp + yq) = yp^2 \qquad \text{..............(1)}$$

Ans : Here $F(x,y,z,p,q) = 2(z + xp + yq) - yp^2$

$$F_x = 2p, \quad F_y = 2q - p^2, \quad F_z = 2, \quad F_p = 2x - 2yp, \quad F_q = 2y$$

Charpit's equations are

$$\frac{dp}{F_x + pF_z} = \frac{dq}{F_y + qF_z} = \frac{dz}{-pF_p - qF_q} = \frac{dx}{-F_p} = \frac{dy}{-F_q}$$

or $\dfrac{dp}{2p+2p} = \dfrac{dq}{2q - p^2 + 2q} = \dfrac{dz}{-p(2x-2yp) - q(2y)} = \dfrac{dx}{-(2x-2yp)} = \dfrac{dy}{-2y}$ ....(2)

From ist and last of the ratios in (2) we get $\dfrac{dp}{4p} = \dfrac{dy}{-2y}$ or $\dfrac{dp}{2p} + \dfrac{dy}{y} = 0$

Integrating $\dfrac{1}{2}\log p + \log y = \text{constant} = \log a$ (say)

or $\quad p^{1/2} y = a \quad$ i.e. $\; p = a/y^2$ ....(3)

We have to solve the equations (1) and (3) for p and q. From equation (1) we have

$$yq = \frac{1}{2}yp^2 - z - xp = \frac{1}{2}y \times \frac{a^2}{y^4} - z - x\frac{a}{y^2}$$

$$\text{or} \quad q = \frac{1}{2}\frac{a^2}{y^4} - \frac{z}{y} - \frac{ax}{y^3}$$

Putting these expression for p and q in dz = pdx + qdy, we get

$$dz = \frac{a}{y^2}dx + \left(\frac{1}{2}\frac{a^2}{y^4} - \frac{z}{y} - \frac{ax}{y^3}\right)dy$$

$$\text{or} \quad dz + \frac{z}{y}dy = a\left(\frac{dx}{y^2} - \frac{xdy}{y^3}\right) + \frac{1}{2}a^2\frac{dy}{y^4}$$

$$\text{or} \quad \frac{ydz + zdy}{y} = \frac{a}{y}\left(\frac{ydx - xdy}{y^2}\right) + \frac{1}{2}a^2\frac{dy}{y^4}$$

$$\text{or} \quad d(yz) = ad\left(\frac{x}{y}\right) + \frac{1}{2}a\frac{dy}{y^3}, \text{ multiplying by y.}$$

Integrating $yz = \frac{ax}{y} + \frac{1}{2}a\frac{y^{-2}}{-2} + b$ or $z = \frac{ax}{y^2} + \frac{b}{y} - \frac{a}{4}\frac{1}{y^3}$

which is the required complete integral with two arbitrary constants a and b.

Ex. 3. Find the complete integral of

$$2(y + zq) = q(xp + yq) \qquad \text{.................. (1)}$$

Solution : Here $F(x,y,z,p,q) = 2(y + zq) - q(xp + yq)$

so that

$$\frac{\partial F}{\partial x} = -qp, \frac{\partial F}{\partial y} = 2 - q^2, \frac{\partial F}{\partial z} = 2q,$$
$$\frac{\partial F}{\partial p} = -xq, \frac{\partial F}{\partial q} = 2z - xp - y2q$$

Charpit's equations are

$$\frac{dp}{F_x + pF_z} = \frac{dq}{F_y + qF_z} = \frac{dz}{-pF_p - qF_q} = \frac{dx}{-F_p} = \frac{dy}{-F_q} \text{ and } dG = 0$$

$$\text{or } \frac{dp}{-qp + p2q} = \frac{dq}{2 - q^2 + q2q} = \frac{dz}{pxq - q(2z - xp - 2yq)} = \frac{dx}{xq} = \frac{dy}{xp + 2yq - 2z} \text{ .... (2)}$$

Equating the 1st and 4th ratios in (2) we get

$$\frac{dp}{pq} = \frac{dx}{xq} \text{ or } \frac{dp}{p} = \frac{dx}{x}$$

Integrating, $\log p = \log x + \text{constant}$ or $p = ax$ ...... (3)

where a is an arbitrary constant. Then (1) gives

$$2y + 2zq = qxax + yq^2$$

$$\text{or} \quad yq^2 + (ax^2 - 2z)q - 2y = 0$$

$$\text{Hence } q = \frac{-(ax^2 - 2z) \pm \sqrt{(ax^2 - 2z)^2 + 4y2y}}{2y}$$

Putting these expressions for p and q in dz = pdx + qdy, we get

$$dz = axdx + \frac{(2z - ax^2) \pm \sqrt{(ax^2 - 2z)^2 + 8y^2}}{2y} dy$$

or $$2ydz = 2axydx + 2zdy - ax^2dy \pm \sqrt{(ax^2 - 2z)^2 + 8y^2}\, dy$$

or $$2(ydz - zdy) + a(x^2dy - 2xydx) = \pm\sqrt{(ax^2 - 2z)^2 + 8y^2}\, dy$$

or $$2y^2 d\left(\frac{z}{y}\right) - ay^2 d\left(\frac{x^2}{y}\right) = \pm\sqrt{(ax^2 - 2z)^2 + 8y^2}\, dy$$

or $$d\left(\frac{2z}{y}\right) - d\left(\frac{ax^2}{y}\right) = \pm\sqrt{\frac{(ax^2 - 2z)^2 + 8y^2}{y^2}}\, dy$$

or $$d\left(\frac{2z - ax^2}{y}\right) = \pm y\sqrt{\left(\frac{(ax^2 - 2z)}{y}\right)^2 + 8} \times \frac{dy}{y^2}$$

or $$\frac{d\left(\frac{2z - ax^2}{y}\right)}{\pm y\sqrt{\left(\frac{(2z - ax^2)}{y}\right)^2 + 8}} = \frac{dy}{y}$$

putting $t = (2z - ax^2)/y$,

or $$\pm\frac{dt}{\sqrt{t^2 + 8}} = \frac{dy}{y}$$

Integrating,

$\pm \log\left(t + \sqrt{t^2 + 8}\right) = \log y + \log b = \log by$ where b is a constant of integration.

or $t + \sqrt{t^2 + 8} = by$ or $t^2 + 8 = (by - t)^2 = t^2 + b^2y^2 - 2byt$

or $b^2y^2 - 2byt = 8$ or $b^2y^2 - 2by \times \frac{2z - ax^2}{y} = 8$

or $b^2y^2 - 2b(2z - ax^2) = 8$

which is the complete integral with two arbitrary constants a and b.

§ 6. 4. Special types of 1st order non-linear p. d. equations

we shall here consider some special types of first order p. d. equations whose solutions may be obtained easily by Charpit's method. These types are often referred to as **standard forms.** These are discussed below with examples.

Type 1. Equations involving only p and q.

For equations of the type $f(p,q) = 0$ ....(1)

We have $f_x = 0, f_y = 0, f_z = 0$.

Then the Charpit's equations reduced to

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{0} = \frac{dq}{0}$$

An obvious solution of these equations is $p = a$ .... (2)

Then the corresponding value of q obtained from equation (1) is given by

$f(a,q) = 0$ .... (3)

so that $q = Q(a)$, which is a constant. Putting these in $dz = pdx + qdy$, we get

$$dz = adx + Q(a)dy$$

$$\therefore z = ax + Q(a)y + b \quad \text{............. (4)}$$

where b is another constant of integration. Thus (4) is the complete integral of the given equation (1).

Instead of choosing the equation $dp = 0$ or $p = a$ to provide our second equation, we may choose $dq = 0$, leading to $q = a$ in some problems.

Ex. 4. Find the complete integral of the equation

$$p + q = pq \quad \text{...............(1)}$$

Solution : Here $F(x,y,z,p,q) = f(p,q) = p + q - pq$

$$f_p = 1 - q, f_q = 1 - p, f_x = 0, f_y = 0, f_z = 0$$

Then Charpit's equations are

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{0} = \frac{dq}{0}$$

We have here $dp = 0$ or $p = a$ .... (2)

Then from (1), $q(p - 1) = p$

or $\quad q = \frac{p}{p-1} = \frac{a}{a-1}$ .... (3)

Putting these values of p and q in

$$dz = pdx + qdy, \quad \text{we get} \quad dz = adx + \frac{a}{a-1}dy$$

Integrating , $z = ax + \frac{a}{a-1}y + b$

This is the complete integral with two arbitrary constants a and b.

Ex. 5. Using the transformation $X = \frac{1}{x}$, $Y = \log y$, $Z = \log z$.

Solve the p. d. equation $x^4p^2 - yzq - z^2 = 0$ ...... (1)

Solution : Given $X = \frac{1}{x}$, $Y = \log y$, $Z = \log z$ or $y = e^Y$, $z = e^Z$

Then $\quad p = \frac{\partial z}{\partial x} = \frac{\partial e^Z}{\partial X}\frac{\partial X}{\partial x} = e^Z = \frac{\partial Z}{\partial X}\left(-\frac{1}{x^2}\right)$

or $\quad x^2p = -e^Z\frac{\partial Z}{\partial X}$

$$q = \frac{\partial z}{\partial y} = \frac{\partial e^Z}{\partial Y} = \frac{\partial Y}{\partial y} = e^Z = \frac{\partial Z}{\partial Y}\frac{1}{y}$$

or $\quad yq = e^Z\frac{\partial Z}{\partial Y}$

The equation (1) now reduces to

$$e^{2Z}\left(\frac{\partial Z}{\partial X}\right)^2 - e^Z\frac{\partial Z}{\partial Y}e^Z - e^{2Z} = 0$$

or $\quad \left(\frac{\partial Z}{\partial X}\right)^2 - \left(\frac{\partial Z}{\partial Y}\right) - 1 = 0$

or $\quad p'^2 - q' - 1 = 0$ .... (2)

where $p' = \frac{\partial Z}{\partial X}, q' = \frac{\partial Z}{\partial Y}$.

The equation (2) is of the form $f(p', q') = 0$.

$$\left.\begin{array}{l}\text{So we take } p' = \text{constant} = a \\ \text{Then(2) gives } q' = p'^2 - 1 = a^2 - 1\end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots (3)$$

Substituting these in the equation $dZ = p'dX + q'dY$, we get

$dZ = adX + (a^2 - 1)dy$.

Integrating, $Z = aX + (a^2 - 1)Y + b$

or $\log z = a\frac{1}{x} + (a^2 - 1)\log y + b$

which is the complete integral of the given equation with two arbitrary constants a and b.

Ex. 6. Find the complete integral of

$$(x^2 + y^2)(p^2 + q^2) = 1 \quad \ldots\ldots\ldots\ldots\ldots (1)$$

Solution : Put $x = r\cos\theta$, $y = \sin\theta$, so that

$$x^2 + y^2 = r^2,\ \theta = \tan^{-1}\left(\frac{y}{x}\right) \quad \ldots\ldots\ldots\ldots\ldots (2)$$

$$\therefore p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial x} \quad \ldots\ldots\ldots\ldots\ldots (3)$$

Now from (2), $r^2 = x^2 + y^2$

$$\therefore 2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{y}{r} = \cos\theta$$

and $$2r\frac{\partial r}{\partial y} = 2y \Rightarrow \frac{\partial r}{\partial y} = \frac{x}{y} = \sin\theta$$

Again $x = r\cos\theta$, differentiating it w.r.t x,

$$1 = \frac{\partial}{\partial x}(r\cos\theta) = \frac{\partial r}{\partial x}\cos\theta + r(-\sin\theta)\frac{\partial \theta}{\partial x}$$

or $$1 = \cos\theta\cos\theta - r\sin\theta\frac{\partial \theta}{\partial x} \Rightarrow -r\sin\theta\frac{\partial \theta}{\partial x} = 1 - \cos^2\theta$$

$$\therefore \frac{\partial \theta}{\partial x} = \frac{\sin^2\theta}{-r\sin\theta} = -\frac{\sin\theta}{r}$$

Again differentiating $y = r\sin\theta$, w.r.t. y,

$$1 = \frac{\partial}{\partial y}(r\sin\theta) = \frac{\partial r}{\partial y}\sin\theta + r\cos\theta\frac{\partial \theta}{\partial y} = \sin\theta\sin\theta + r\cos\theta\frac{\partial \theta}{\partial y}$$

or $\quad r\cos\theta\dfrac{\partial\theta}{\partial y}=1-\sin^2\theta=\cos^2\theta$

$$\therefore \frac{\partial\theta}{\partial y}=\frac{\cos\theta}{r}$$

On using these results we get from (3),

$$p=\frac{\partial z}{\partial r}\cos\theta+\frac{\partial z}{\partial\theta}\left(\frac{-\sin\theta}{r}\right)$$

and $\quad q=\dfrac{\partial z}{\partial y}=\dfrac{\partial z}{\partial r}\dfrac{\partial r}{\partial y}+\dfrac{\partial z}{\partial\theta}\dfrac{\partial\theta}{\partial y}$

$$=\frac{\partial z}{\partial r}\sin\theta+\frac{\partial z}{\partial\theta}\left(\frac{\cos\theta}{r}\right)$$

On putting these expressions for p and q, the equation (1) becomes

$$r^2\left[\left\{\left(\frac{\partial z}{\partial r}\right)^2\cos^2\theta+\left(\frac{\partial z}{\partial\theta}\right)^2\frac{\sin^2\theta}{r^2}-2\frac{\cos\theta\sin\theta}{r}\frac{\partial z}{\partial r}\frac{\partial z}{\partial\theta}\right\}\right.$$
$$\left.+\left\{\left(\frac{\partial z}{\partial r}\right)^2\sin^2\theta+\left(\frac{\partial z}{\partial\theta}\right)^2\frac{\cos^2\theta}{r^2}-2\frac{\sin\theta\cos\theta}{r}\frac{\partial z}{\partial r}\frac{\partial z}{\partial\theta}\right\}\right]=1$$

or $\quad r^2\left[\left(\dfrac{\partial z}{\partial r}\right)^2+\left(\dfrac{\partial z}{\partial\theta}\right)^2\dfrac{1}{r^2}\right]=1$

or $\quad \left(r\dfrac{\partial z}{\partial r}\right)^2+\left(\dfrac{\partial z}{\partial\theta}\right)^2=1 \qquad \text{.................. (4)}$

Now put log r = R on $r=e^R$,

so that $\dfrac{\partial R}{\partial r}=\dfrac{1}{r}$ and $\dfrac{\partial z}{\partial r}=\dfrac{\partial z}{\partial R}\dfrac{\partial R}{\partial r}=\dfrac{1}{r}\dfrac{\partial z}{\partial R}\Rightarrow r\dfrac{\partial}{\partial r}=\dfrac{\partial}{\partial R}$

Then equation (4) becomes $\quad \left(\dfrac{\partial z}{\partial R}\right)^2+\left(\dfrac{\partial z}{\partial\theta}\right)^2=1 \qquad \text{........... (5)}$

This is of the form f (p,q) = 0.We choose $p=\dfrac{\partial z}{\partial R}=a$. Then from (5),

$$q=\frac{\partial z}{\partial\theta}=\sqrt{1-\left(\frac{\partial z}{\partial R}\right)^2}=\sqrt{1-a^2}$$

Putting these in $dz = pdR + qd\theta$, we get $dz = adR + \sqrt{1-a^2}\,d\theta$.

Integrating $z = aR + \sqrt{1-a^2}\,\theta + c$

$$= a\log r + \sqrt{1-a^2}\,\theta + c \qquad \ldots\ldots\ldots\ldots\ldots (6)$$

where $r = \sqrt{x^2+y^2}$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ and a and c are arbitrary constants.

So equation (6) is the complete integral of the given p. d. equation.

Type II. Equations not involving the independent variables

If the partial differential equation is of the form $f(z,p,q) = 0$ .... (1)

So that $f_x = 0$, $f_y = 0$, then the Charpit's equation reduces to

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{-pf_z} = \frac{dq}{-qf_z} \qquad \ldots\ldots\ldots\ldots\ldots (2)$$

The last two ratios of (2) gives $\frac{dp}{p} = \frac{dq}{q}$ i.e. $q = \alpha p$ .... (3)

Solving (1) and (3), we get p and q in terms of z, say $p = \phi(z)$, $q = \alpha p = \alpha\phi(z)$

Then $dz = pdx + qdy$ gives $dz = \phi(z)dx + \alpha\phi(z)dy$

or, $\int \frac{dz}{\phi(z)} + \beta = x + \alpha y$ ....(4) where α and β are arbitrary constants.

It appears that, in the complete integral (4), z will be a function of the combination $x + \alpha y$. So the solution may be obtained by the following procedure:-

Let $z = g(x + \alpha y) = g(u)$, $u = x + \alpha y$ ....(5)

so that

$$p = \frac{\partial z}{\partial x} = \frac{\partial g}{\partial x} = \frac{dg}{du}\frac{\partial u}{\partial x} = \frac{dg}{du} = \frac{dz}{du}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial g}{\partial y} = \frac{dg}{du}\frac{\partial u}{\partial y} = \frac{dg}{du}\alpha = \alpha\frac{dz}{du}$$

Then the partial differential equation (1) is transformed to the ordinary differential equation of the first order :

$$f\left(z, \frac{dz}{du}, \alpha\frac{dz}{du}\right) = 0 \qquad \ldots\ldots\ldots\ldots (6)$$

the general solution of equation (6) will contain the arbitrary constant α and another constant of integration β, and this will give the complete integral of the p.d. equation (1).

Ex 7. Find the complete solution of the p.d. equation

$$z^2(p^2+q^2+1)=1 \qquad \text{..........(1)}$$

and a singular integral.

Solution : The equation is of the form f(z, p, q) = 0.

We put $z = g(u), \quad u = x + \alpha y$

so that

$$p=\frac{\partial z}{\partial x}=\frac{dg}{du}\frac{\partial u}{\partial x}=\frac{dg}{du}=\frac{dz}{du}$$
$$q=\frac{\partial z}{\partial y}=\frac{dg}{du}\frac{\partial u}{\partial y}=\alpha\frac{dz}{du}$$

Then the given equation (1) reduces to

$$z^2\left\{\left(\frac{dz}{du}\right)^2+\alpha^2\left(\frac{dz}{du}\right)^2+1\right\}=1$$

or, $$(1+\alpha^2)\left(\frac{dz}{du}\right)^2=\frac{1}{z^2}-1=\frac{1-z^2}{z^2}$$

or, $$\sqrt{1+\alpha^2}\,\frac{zdz}{\sqrt{1-z^2}}=\pm du$$

put $t = 1 - z^2$
$dt = -2zdz$

$$\therefore \int\frac{zdz}{\sqrt{1-z^2}}=\int\frac{-\frac{1}{2}dt}{t^{\frac{1}{2}}}$$

$$=-\frac{1}{2}\frac{t^{\frac{1}{2}}}{\frac{1}{2}}=-t^{\frac{1}{2}}=-(1-z^2)^{\frac{1}{2}}$$

Integrating,

$$\sqrt{1+\alpha^2}\sqrt{1-z^2}=\pm(u+\beta)$$

Squaring we get

$$(1+\alpha^2)(1-z^2)=(x+\alpha y+\beta)^2 \qquad \text{....(2)}$$

This is the complete integral with α and β as arbitrary constants.

Singular Integral:

The equation (2) is

$$f(x, y, z, \alpha, \beta) = (1+\alpha^2)(1-z^2)-(x+\alpha y+\beta)^2=0$$

Differentiating this with respect to α and β, partially, we get,

$$\frac{\partial f}{\partial \alpha} = 0 \Rightarrow 2\alpha(1 - z^2) - 2(x + \alpha y + \beta)y = 0 \quad \text{.......(3)}$$

and $$\frac{\partial f}{\partial \beta} = 0 \Rightarrow -2(x + \alpha y + \beta).1 = 0 \quad \text{.......(4)}$$

From this two equations (3) and (4) we see that

$$z^2 - 1 = 0 \quad z = \pm 1 \quad \text{......(5)}$$

and that these values of z along with (4) satisfy the equation (2).

Thus the two planes $z = \pm 1$, give the singular solution of the p.d. equation (1), that is, the envelope of the surfaces represented by the complete integral (2).

Ex 8. Solve by Charpit's method

$$16p^2 z^2 + 9q^2 z^2 + 4z^2 - 4 = 0 \quad \text{.......(1)}$$

Solution: The given equation is of the form

$$f(z, p, q) = 0 \quad \text{........(2)}$$

So we choose $p = aq$ ....(3)

Then (1) gives $16a^2q^2z^2 + 9q^2z^2 = 4(1-z^2)$

$$\text{or,} \quad q^2 = \frac{4}{(16a^2 + 9)} \times \frac{1 - z^2}{z^2}$$

$$\text{or,} \quad q = \pm \frac{2}{\sqrt{16a^2 + 9}} \times \frac{\sqrt{1 - z^2}}{z} \quad \text{.......(4)}$$

Substituting these expressions for p and q in

$dz = pdx + qdy$, we get (positive sign only),

$$dz = \frac{2a}{\sqrt{16a^2 + 9}} \times \frac{\sqrt{1 - z^2}}{z} dx + \frac{2}{\sqrt{16a^2 + 9}} \frac{\sqrt{1 - z^2}}{z} dy$$

$$\text{or,} \quad \frac{zdz}{\sqrt{1 - z^2}} = \frac{2a}{\sqrt{16a^2 + 9}} dx + \frac{2}{\sqrt{16a^2 + 9}} dy$$

Integrating

$$\int \frac{zdz}{\sqrt{1 - z}} = \frac{2}{\sqrt{16a^2 + 9}} (ax + y) + \text{constant} \quad \text{....(5)}$$

Now putting $1 - z^2 = t$, so that $-2zdz = dt$, L.H.S of (5) is $= \int -\frac{1}{2}\frac{dt}{t^{\frac{1}{2}}} = -\frac{1}{2}\frac{t^{\frac{1}{2}}}{\frac{1}{2}} = -t^{\frac{1}{2}} = -\sqrt{1-z^2}$

Hence (5) becomes

$$-\sqrt{1-z^2} = \frac{2}{\sqrt{16a^2+9}}(ax+y) + \text{constant}$$

or $\quad -\sqrt{1-z^2}\sqrt{16a^2+9} = 2(ax+y)+b$

squaring, $\quad (16a^2+9)(1-z^2) = \{2(ax+y)+b\}^2$

This is the complete integral with two arbitrary constants a and b.

Ex 9. Find the complete integral and the singular integral of the p.d. equation

$$q^2 = z^2p^2(1-p^2) \qquad \ldots\ldots(1)$$

Solution : The given p.d. equation is of the form $\quad f(z, p, q) = 0$

So we take $\quad q = ap \qquad \ldots\ldots (2)$

From (1) and (2) we have $\quad a^2p^2 - z^2p^2(1-p^2) = 0$ or, $p^2(a^2 - z^2 + z^2p^2) = 0$

$\therefore$ either $p = 0$ or $z^2p^2 = z^2 - a^2$ i.e. $z = \pm\frac{\sqrt{(z^2-a^2)}}{z}$

Now $dz = pdx + qdy = p(dx + ady) \qquad \ldots.(3)$

when $p = 0$, (3) gives $dz = 0 \therefore z = C \qquad \ldots\ldots(4)$

when $p = \pm\frac{\sqrt{(z^2-a^2)}}{z}$, (3) gives

$$dz = \pm\left\{\frac{\left(\sqrt{z^2+a^2}\right)}{z}\right\}(dx+ady)$$

or, $\quad dx + ady = \pm\frac{1}{2}\left(z^2-a^2\right)^{-\frac{1}{2}}(2z)dz = \pm\frac{1}{2}\left(z^2-a^2\right)^{-\frac{1}{2}}d\left(z^2-a^2\right)$

Integrating,

$$x + ay + b = \pm\frac{1}{2}\frac{\left(z^2+a^2\right)^{\frac{1}{2}}}{\frac{1}{2}} \qquad \ldots.(5)$$

Squaring it, $(x + ay + b)^2 = (z^2 - a^2)$

Hence the required complete integral is

$$z^2 - a^2 = (x + ay + b)^2 \text{ or } z = C \qquad \text{....(6)}$$

Singular Integral

On differentiating partially with respect to a and b, the first equation of (6) gives

$$-2a = 2(x + ay + b)\, y \qquad \text{.....(7)}$$

and $\quad 0 = 2(x + ay + b)\, 1 \qquad$ .....(8)

From (7) and (8), $x + ay + b = 0$ and $a = 0$.

Putting these values in the first equation of (6), we get $z = 0$.

Again for $z = 0$, $p = \dfrac{\partial z}{\partial x} = 0$ and $q = \dfrac{\partial z}{\partial y} = 0$.

These values of p and q clearly satisfies the given differential equation (1). Hence $z = 0$ is the required singular integral.

Further $z = 0$ is a particular case of $z = C$ in (6) corresponding to $C = 0$. Hence the plan $z = 0$ is the singular integral as well as a particular solution of the complete integral.

Type III. Separable Equation : f(x,p) = g(y,q)

If the given non-linear first order p. d. equation can be put in the form

$$f(x,p) = g(y,q) \qquad \text{................ (1)}$$

so that $F(x,y,z,p,q) = f(x,p) - g(y,q)$

Then $\quad F_x = f_x,\ F_y = -g_y,\ F_z = 0,\ F_p = f_p,\ F_q = -g_q$

The Charpit's equations become

$$\frac{dx}{f_p} = \frac{dy}{-g_q} = \frac{dz}{pf_p - qg_q} = \frac{d_p}{-f_x} = \frac{d_q}{g_y} \qquad \text{............... (2)}$$

From the first and the fourth of these ratios we have

$$f_x dx + f_p dp = 0 \qquad \text{............... (3)}$$

and from the second and the fifth,

$$g_y dy + g_q dq = 0 \qquad \text{............... (4)}$$

But the L.H.S of the (3) and (4) are the total differentials df and dg of the functions f (x,p) and g (y,q). Hence the integrals of (3) and (4) are

$f(x,p) = \text{constant}$

and $g(y,q) = \text{constant}.$

Again by the original equation (1), we see that these two constants are equal. Hence $f(x,p) = \alpha$, $g(y,q) = \alpha$ ......(5)

Thus, to solve the given equation (1) we proceed by solving the equation (5) for p in terms of x and α and for q in terms of y and α. Then putting these expressions in

$$dz = p(x,\alpha)dx + q(y,\alpha)dy$$

and integrating, we obtain the complete integral of the given equation.

Ex. 10. Find the complete integral of the p. d. equation

$$p^2q^2 + x^2y^2 = x^2q^2(x^2 + y^2)$$

Solution : The given equation can be rearranged as

$$\frac{p^2}{x^2} + \frac{y^2}{q^2} = x^2 + y^2$$

or $$\frac{p^2}{x^2} - x^2 = y^2 - \frac{y^2}{q^2} = y^2\left(\frac{q^2 - 1}{q^2}\right) \quad \text{.................. (1)}$$

This is of the form $f(x,p) = g(y,q)$

We put $\frac{p^2}{x^2} - x^2 = a^2$ or $p^2 = (a^2 + x^2)x^2$.

i.e. $p = x\sqrt{a^2 + x^2}$ .... (2)

and $y^2\left(1 - \frac{1}{q^2}\right) = a^2$ or $1 - \frac{1}{q^2} = \frac{a^2}{y^2}$ or $q^2 = \frac{y^2}{y^2 - a^2}$

i.e. $q = \frac{y}{\sqrt{y^2 - a^2}}$ .... (3)

Putting these value of p and q in $dz = pdx + qdy$, we get

$$dz = x\sqrt{x^2 + a^2}dx + \frac{y}{\sqrt{y^2 - a^2}}dy$$

Integrating,

$$z=\int\sqrt{x^2+a^2}\times\frac{1}{2}d(x^2+a^2)+\int\frac{\frac{1}{2}d(y^2-a^2)}{\sqrt{y^2-a^2}}+b$$

$$=\frac{1}{2}\times\frac{(x^2+a^2)^{3/2}}{3/2}+\frac{1}{2}\frac{(y^2-a^2)^{1/2}}{1/2}+b$$

or $$z=\frac{1}{3}(x^2+a^2)^{3/2}+(y^2-a^2)^{1/2}+b.$$

This is the complete integral of the given equation.

Ex. 11. Find the complete integral of the equation

$$z(p^2-q^2)=x-y$$

Solution : The given equation is $\left(\sqrt{z}\frac{\partial z}{\partial x}\right)^2-\left(\sqrt{z}\frac{\partial z}{\partial y}\right)^2=x-y$ ....(1)

Let us put $\sqrt{z}dz=dz$ or $z^{3/2}\times\frac{2}{3}=Z$

So that

$$P=\frac{\partial Z}{\partial x}=\frac{\partial Z}{\partial z}\frac{\partial z}{\partial x}=\sqrt{z}p$$
$$Q=\frac{\partial Z}{\partial y}=\frac{\partial Z}{\partial z}\frac{\partial z}{\partial y}=\sqrt{z}q$$

Then equation (1) becomes

$$P^2-Q^2=x-y$$

or $$P^2-x=Q^2-y$$

which is of of the form f (x,p) = g (y,q).

So we have each side of the last equation is a constant C, say

$$\therefore\quad P^2-x=c\Rightarrow P=\sqrt{x+c}$$
and $$Q^2-y=c\Rightarrow Q=\sqrt{y+c}$$

Then $dZ=Pdx+Qdy=(x+c)^{1/2}dx+(y+c)^{1/2}dy$

Integrating,

$$Z = (x+c)^{3/2} \times \frac{2}{3} + (y+c)^{3/2} \times \frac{2}{3} + \text{constant}$$

or $$\frac{2}{3} z^{3/2} = \frac{2}{3}(x+c)^{3/2} + \frac{2}{3}(y+c)^{3/2} + \text{constant}$$

or $$z^{3/2} = (x+c)^{3/2} + (y+c)^{3/2} + b \qquad \text{................(2)}$$

where a and b are arbitrary constants. Equation (2) is the complete integral of the given p. d. equation.

Ex. 12. Find the complete integral of $z(xp - yq) = y^2 - x^2$.

Put $zdz = dZ$ or $Z = \frac{1}{2}z^2$, then

$$P = \frac{\partial Z}{\partial x} = \frac{\partial Z}{\partial z}\frac{\partial z}{\partial x} = zp \qquad Q = \frac{\partial Z}{\partial y} = \frac{\partial Z}{\partial z}\frac{\partial z}{\partial y} = zq$$

The given equation now becomes

$$xP\ yQ = y^2 - x^2$$

or $$xP + x^2 = yQ + y^2 \qquad \text{................ (1)}$$

This is of the form f (x,p) = g (y,q).

So taking each side of (1) equal to the same constant C, say, we get

$$xP + x^2 = C \text{ and } yQ + y^2 = C$$

or $$P = \frac{c - x^2}{x}, Q = \frac{c - y^2}{y}$$

Then $dZ = Pdx + Qdy$ gives

$$dZ = \left(\frac{c}{x} - x\right)dx + \left(\frac{c}{y} - y\right)dy$$

Integrating,

$$Z = c\log x - \frac{1}{2}x^2 + c\log y - \frac{1}{2}y^2 + \text{constant}$$

or $$\frac{1}{2}z^2 = c\log(xy) - \frac{1}{2}(x^2 + y^2) + \text{constant}$$

or $$z^2 = 2c\log(xy) - (x^2 + y^2) + b$$

which is the complete integral.

Type 4. Clairaut Equations

A first order partial differential equation is said to be of Clairaut type if it can be written in the form

$$z = px + qy + f(p,q) \qquad \ldots\ldots\ldots\ldots\ldots (1)$$

Here $F(x,y,z,p,q) = xp + yq + f(p,q) - z$

$$F_x = p, F_y = q, F_z = -1, F_p = x + f_p, F_q = y + f_q$$

The corresponding Charpit's equations are

$$\frac{dx}{x+f_p} = \frac{dy}{y+f_q} = \frac{dz}{px+q+pf_p+qf_q} = \frac{dp}{0} = \frac{dq}{o}$$

so, $dp = 0, dq = 0.$

So we may take either p = constant = α or q = constant = β in conjunction with the given equation (1).

However, both these integrals are used and the equation (1) can be written as

$$z = \alpha x + \beta y + F(\alpha,\beta) \qquad \ldots\ldots\ldots\ldots\ldots (2)$$

This realtion (2) yields $p = \alpha$ and $q = \beta$ and it has two arbitrary constants α and β. So it is the complete integral of the p. d. equation (1).

Usually the family of plans (2) will posses an envelope which will be the singular solution of (1).

Ex. 13. Solve and find the singular solution of the p. d. equation

$$z = px + qy + p^2q^2 \qquad \ldots\ldots\ldots\ldots\ldots (1)$$

Solution : The equation (1) is of Cairaut type. So putting $p = a$ and $q = b$ in (1) we get the complete integral as

$$z = ax + by + a^2b^2 \qquad \ldots\ldots\ldots\ldots\ldots (2)$$

For singular solution, we differentiate equation (2) w.r.t. a and b, partially, in turn to get

$$0 = x + 2ab^2 \text{ or } 2ab^2 = -x \qquad \ldots\ldots\ldots\ldots (i)$$

and $$0 = y + 2a^2b \text{ or } 2a^2b = -y \qquad \ldots\ldots\ldots\ldots (ii)$$

Squaring these, $4a^2b^4 = x^2$ $\qquad \ldots\ldots\ldots\ldots (iii)$

and $4a^4b^2 = y^2$ $\qquad \ldots\ldots\ldots\ldots (iv)$

Now (iv)/(i) $\Rightarrow 2a^3 = \frac{-y^2}{x} = (-1)^3 \frac{y^2}{x}$

or $\quad a^3 = (-1)^3 \frac{y^2}{2x} \Rightarrow a = -\left(\frac{y^2}{2x}\right)^{\frac{1}{3}}$

Similarly by (iii)/(ii) we get $\quad b = -\left(\frac{x^2}{2y}\right)^{\frac{1}{3}}$

Substituting these expressions for a and b in (2) we obtain the singular solution the p. d. equation (1) as

$$z = -x\left(\frac{y^2}{2x}\right)^{\frac{1}{3}} - y\left(\frac{x^2}{2y}\right)^{\frac{1}{3}} + \left(\frac{x^2y^2}{16}\right)^{\frac{1}{3}}$$

Ex. 14. Find the singular integral of the p.d. equation

$$xp^2q + ypq^2 - zpq + 1 = 0 \quad \text{.................. (1)}$$

Also find a developable surface belonging to the general integral.

Solution : The given equation (1) can be written as

$$zpq = xp^2q + ypq^2 + 1$$

or $\quad z = xp + qy + \frac{1}{pq} \quad$ .... (2)

This is of the form $z = xp + yq + f(p,q)$

Hence its solution can be taken as

$$z = \alpha x + \beta y + \frac{1}{\alpha\beta} \quad \text{................ (3)}$$

where α, β are two arbitrary constants.

Singular integral:

Differentiating (3) partially w.r.t α and β in turn we get

$$0 = x - \frac{1}{\alpha^2\beta} \text{ i.e. } \frac{1}{\alpha^2\beta} = x \quad \text{.................(i)}$$

and $\quad 0 = \beta - \frac{1}{\alpha\beta^2}$ i.e. $\frac{1}{\alpha\beta^2} = y \quad$ ....(ii)

Now (ii)/(i) gives

$$\frac{\alpha}{\beta}=\frac{y}{x}\Rightarrow\alpha=\frac{y}{x}\beta \qquad \text{...............(iii)}$$

$$\text{Then (i)}\Rightarrow\beta-\frac{1}{\alpha^2 x}=\frac{1}{x}\frac{x^2}{y^2\beta^2}\ \text{ or }\ \beta^3=\frac{x}{y^2}\ \text{ i.e. }\ \beta=\left(\frac{x}{y^2}\right)^{\frac{1}{3}} \qquad \text{....... (iv)}$$

$$\text{Then(iii)}\Rightarrow\alpha=\frac{y}{x}\beta=\frac{y}{x}\left(\frac{x}{y^2}\right)^{\frac{1}{3}}=\left\{\frac{y^3}{x^3}\frac{x}{y^2}\right\}^{\frac{1}{3}}\ \text{ i.e. }\ \alpha=\left(\frac{y}{x^2}\right)^{\frac{1}{3}} \qquad \text{....... (v)}$$

Substituting these expressions for α and β in (3) we get

$$z=x\left(\frac{y}{x^2}\right)^{\frac{1}{3}}+y\left(\frac{x}{y^2}\right)^{\frac{1}{3}}+\left(\frac{x^2}{y}\right)^{\frac{1}{3}}\left(\frac{y^2}{x}\right)^{\frac{1}{3}}$$

or $$z=(xy)^{\frac{1}{3}}+(xy)^{\frac{1}{3}}+(xy)^{\frac{1}{3}}=3(xy)^{\frac{1}{3}}$$

$$z^3=27\,xy \qquad \text{................ (4)}$$

From the equation (4) we get, $3z^2p=27y$, $3z^2q=27x$. Putting these in (2) we obtain

$$z=\frac{27yx}{3z^2}+\frac{27xy}{3z^2}+\frac{3z^2}{27y}\times\frac{3z^2}{27x}$$

or $$z=\frac{z^3}{3z^2}+\frac{z^3}{3z^2}+\frac{9z^4}{27\times z^3}$$ or $$z=\frac{1}{3}z+\frac{1}{3}z+\frac{1}{3}z=z$$

Thus the equation (2) is satisfied by (4). Hence the surface (4) is the singular integral of the given p.d. equation (2).

Developable surface belonging to the general integral

Let us choose $\beta=g(\alpha)=\frac{1}{\alpha}$, so that the complete integral (3) has the one-parameter

subsystem $$z=\alpha x+\frac{1}{\alpha}y+1 \qquad \text{.............. (5)}$$

Differentiating this partially w.r.t α,

$$0=x-\frac{1}{\alpha^2}y\ \text{ or }\ \frac{1}{\alpha^2}=\frac{x}{y}\quad \text{i.e}\quad \alpha^2=\frac{y}{x}\ \text{ or }\ \alpha=\sqrt{\frac{y}{x}}$$

Then (5) becomes $z-1=\sqrt{\frac{y}{x}}x+\sqrt{\frac{x}{y}}y=\sqrt{xy}+\sqrt{xy}=2\sqrt{xy}$

or $(z-1)^2=4xy$ .... (6)

Equation (5) is a general integral of the p. d. equation (2) and represents a system of planes for various values of the parameter α.

Equation (6) represents the envelope of the planes (5) and so is a developable surface belonging to the general integral (5).

Exercise

1. Find the complete integrals of the following partial differential equations

(i) $(p^2+q^2)y=qz$ Ans : $z^2-\alpha^2y^2=(\alpha x+\beta)^2$, α, β constants.

(ii) $p=(z+qy)^2$ Ans : $yz=ax+2\sqrt{ay}+b$, a, b constants.

(iii) $z^2(p^2z^2+q^2)=1$ Ans : $(z^2+a^2)^3=q(x+ay+b)^2$, a,b constants.

(iv) $2xz-px-2qxy+pq=0$ Ans : $z-ay=b(x^2-a)$, a,b constants.

(v) $p^2+q^2-2px-2qy+1=0$

Ans : $(a^2+1)z=\frac{1}{2}v^2\pm\frac{1}{2}v\sqrt{\{v^2-(a^2+1)\}}$

$$-\frac{1}{2}(a^2+1)\left[v+\sqrt{\{v^2-(a^2+1)\}}\right]+b \quad \text{where } v=ax+b.$$

2. (i) Find the complete integral of the equation $x^2p^2+y^2q^2=z^2$

Ans : $\log z=a\log x+b\log y+c, b=\sqrt{1-a^2}$

(ii) Find the complete integral of $(y-x)(qy-px)=(p-q)^2$

Hints : Take $x+y=X$, $xy=Y$. Ans : $z=a(x+y)+\sqrt{a}xy+c$

(iii) Find the complete integral of $p^2+q^2=npq$.

Ans : $z=ax+\frac{1}{2}a\left[n\pm\sqrt{n^2-4}\right]y+c$

(iv) Using the transformation $X=\log x, Z=\frac{1}{2}z^2$, solve the equation

$z^2(p^2x^2+q^2)=1$ Ans : $z^2=2\alpha\log x+2\sqrt{1-\alpha^2}y+\beta$

3. (i) Solve the p. d. equation $x^2p^2 + y^2q^2 = z$ Ans : $4(1+a^2)z = (a\log x + \log y + b)^2$

(ii) Find the complete integral of $p^2 = z^2(1 - pq)$.

Ans : $\pm(x + ay) = b - \sinh^{-1}\left(\frac{1}{z\sqrt{a}}\right) + \sqrt{1 + az^2}$.

(iii) Solve $p^3 + q^3 = 27z$

Ans : $8(x + ay + b)^2 = (1 + a^2)z^2$ is the complete integral singular integral is $z = 0$.

4. (i) Solve by Charpit's method $q = p^2 - xp$

Ans : $z = \frac{1}{4}\left(x^2 + x\sqrt{x^2 + 4a}\right) + a\log\left\{x + \sqrt{x^2 + 4a}\right\} + ay + b$.

(ii) Find the complete integral of the equation $p^2y(1 + x^2) = qx^2$.

Ans : $z = a\sqrt{1 + x^2} + \frac{1}{2}a^2y^2 + b$

(iii) Solve $z^2(p^2 + q^2) = x^2 + y^2$

Ans : $z^2 = x\sqrt{(c^2 + x^2)} + c^2\log\left\{x + \sqrt{(c^2 + x^2)}\right\} + y\sqrt{(y^2 - c^2)}$

$$- c^2\log\left\{y + \sqrt{(y^2 - c^2)}\right\} + \text{constant}.$$

(iv) Find the complete integral of $2x(z^2q^2 + 1) = pz$

Ans : $z^2 = 2x^2(1 + a) + 2\sqrt{a}y + 2b$.

5. (i) Find the complete integral and singular solution of the p. d. equation

$$z = px + qy + c\sqrt{(1 + p^2 + q^2)}$$

Ans : Complete int is $z = ax + by + c\sqrt{(1 + a^2 + b^2)}$ and singular integral is $x^2 + y^2 + z^2 = c^2$

(ii) Find the singular integral of $z = px + qy - 2\sqrt{(pq)}$. Ans : $xy = 1$.

(iii) Find the complete integral and the singular integral of the equation

$$z = px + qy + p^2 + q^2$$

Ans : C.I. is $z = ax + by + a^2 + b^2$ S.I. is $4z + x^2 + y^2 = 0$.

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