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Institute of Distance and Open Learning

Gauhati University

M.A./M.Sc. in Economics

Semester II

Paper VII

Mathematical Methods for Economic Analysis-II

Contents:

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MA/M.Sc. Economics
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COURSE STRUCTURE

A student shall do a total number of sixteen papers in the four Semesters. Each paper will carry 100 marks - 20 marks for internal evaluation during the semester and 80 marks for external evaluation through end semester examination. All the papers in the First, Second and Third Semesters will be compulsory. The paper XIII and XIV of the Fourth Semester will also be compulsory. The remaining two papers for the Fourth Semesters will be chosen by a student from the optional papers. The names and numbers assigned to the papers are as follows.

First Semester

- I Microeconomics Theory
- II Macroeconomics Theory - I
- III Mathematical Methods for Economic Analysis-I
- IV Statistical Methods for Economic Analysis

Second Semester

- V Advanced Microeconomics
- VI Macroeconomic Theory -II
- VII Mathematical Methods for Economic Analysis-II
- VIII Elementary Econometrics

Third Semester

- IX Development Economics-I
- X International Economics
- XI Issues in Indian Economy
- XII Public Finance-I

Fourth Semester

- XIII Development Economics-II
- XIV Public Finance-II

Papers XV and XVI are optional

A student has to choose any two of the following courses.

- (a) Population and Human Resource Development
- (b) Econometric Methods
- (c) Environmental Economics
- (d) Financial System

Detailed Contents of this Paper

Paper - VII

MATHEMATICAL METHODS FOR ECONOMIC ANALYSIS-II

Unit – 1: Optimization with Equality Constraint

Optimization with equality contraints, Lagrange's multiplier method–application to consumer's equilibrium and producer's equilibrium in factor market.

Unit – 2: Calculus for Dynamic Analysis

First order differential equation and its solutions – application to dynamic stability of market and simple growth process (Harrod-Domar), First order difference equation and its solution- application of difference equation – lagged market model (Cobweb) and Harrod's model of growth.

Unit – 3: Optimization with Inequality Constraint

Liner programming, General formulation Transportation problem, diet problem and production problem – Simplex method of solution (two variables, two constraints only) – Concept of duality.

Unit – 4: Game Theory

Two-person Zero sum game – pure strategies with saddle point, games without saddle point – the rules of dominance – solution of games without saddle point – mixed strategies, Basic ideas and examples of non zero sum games – Nash equilibrium, Prisoner's dilemma and Repeated games.

* * *

UNIT 1

OPTIMIZATION WITH EQUALITY CONSTRAINT

Structure :

1.1. Introduction

In our previous chapters we have studied two different contents - optimisation with a single explanatory variable & with two explanatory variables. But these techniques were unconstrained extrema in the sense that the decision made regarding one variable does not impinge upon the choices of the remaining variables. For instance, a two product firm can choose any value for Q_1 & any value for Q_2 it wishes without the two choices limiting each other.

But in practice, the decision of firms is restricted by many economic & other problems. For example, in the theory of consumer behaviour a consumer has to maximise utility keeping in view his total income or budget of the consumer as a constraint. So the budget limitation will restrict the choices of the consumer in deciding the purchase of the basket of goods. Similarly in case of a producer the restriction may come in the form of cost minimisation with the production function as constraint. The primary purpose of imposing a constraint is to give due cognizance to certain limiting factors present in the optimisation problem.

1.2. Objectives :

The main objectives of this chapter are —

1) Analyse the process of constrained optimisation.

2) Application of constrained optimisation process in profit maximisation.

3) Study the process of constrained optimisation in the case of cost minimisation.

1.3 Lagrange Multiplier Method

Lagrange multiplier method is used to convert a constraint extremum problem into a form such that the first order condition on the free extremum problem still can be applied.

Let us consider a function having two explanatory variables x_1, x_2 such that

$$y = f(x_1, x_2) \text{ --------------- } (1)$$

We need to find out the combination of x_1, x_2 that will either maximise or minimise the function subject to the satisfaction of an equality constraint.

$$g(x_1, x_2) = C \text{ --------------- } (2)$$

where C is a constant.

Equation (1) is called the "objective function" & (2) is called an equality constraint.

When we want to optimise the function with the given constraint, the lagrange function will be

$$L = f(x_1, x_2) + \lambda[C - g(x_1, x_2)] \text{ ------- } (3)$$

The symbol λ (Lambda) is called a Lagrange multiplier. Since the Lagrange function given by (3) is now a function of three variables $x_1, x_2 \& \lambda_2$ the maximisation of L requires to satisfy the first order & second order conditions of maximisation.

$$\frac{\partial L}{\partial x_1} = 0, \quad \frac{\partial L}{\partial x_2} = 0 \ \& \ \frac{\partial L}{\partial \lambda} = 0$$

Now, $\frac{\partial L}{\partial x_1} = L_1 = f_1 - \lambda g_1 = 0 \Rightarrow \lambda = \frac{f_1}{g_1}$ -----------(4)

$$\frac{\partial L}{\partial x_2} = L_2 = f_2 - \lambda g_2 = 0 \Rightarrow \lambda = \frac{f_2}{g_2} \text{ --------(5)}$$

$$\frac{\partial L}{\partial \lambda} = C - g(x_1, x_2) = 0$$

From (4) & (5) $\quad \lambda = \frac{f_1}{g_1} = \frac{f_2}{g_2}$

The first order condition can be derived by using total differential approach also

$$y = f(x_1, x_2)$$

$$\Rightarrow dy = f_1 dx_1 + f_2 dx_2 \Rightarrow \frac{dx_1}{dx_2} = \frac{f_2}{f_1} \text{ -------------------(6)}$$

Now let us take total differential of

$$g(x_1, x_2) = C$$

$$\Rightarrow g_1 dx_1 + g_2 dx_2 = d(C) = 0 \Rightarrow \frac{dx_1}{dx_2} = \frac{g_1}{g_2} \text{ ----------(7)}$$

From (6) & (7) $\quad -\frac{f_2}{f_1} = -\frac{g_2}{g_1} \ \Rightarrow \frac{f_1}{g_1} = \frac{f_2}{g_2}$

Second order condition

The sufficient second order condition can be derived with the help of second order differential d^2L. But for simplicity, we will just mention these conditions with the help of a new determinant value $|\overline{H}|$ called Bordered **Hessian determinant.**

For two variables case

$|\overline{H}_2| > 0$, for maximisation & $|\overline{H}_2| < 0$, for minimisation

This condition can be extended for three variables case as follows —

Let us take the objective function with variables x_1, x_2 & x_3

$f = (x_1, x_2, x_3)$ subject to $g(x_1, x_2, x_3) = C$

with $L = f(x_1, x_2, x_3) + \lambda[c - g(x_1, x_2, x_3)]$

The first order necessary conditions

$$L_1 = L_2 = L_3 = 0$$

Second order sufficient condition

The first order condition retmains the same for both maximisation & minimisation. But the second order condition differ from each other.

For minimization the conditions are as follows

$$|\overline{H}_2| < 0, \quad |\overline{H}_3| < 0$$

For maximisation $|\overline{H}_2| > 0$, $|\overline{H}_3| < 0$

Here $|\overline{H}_2| = \begin{vmatrix} 0 & g_1 & g_2 \\ g_1 & L_{11} & L_{12} \\ g_2 & L_{21} & L_{22} \end{vmatrix}$ & $|\overline{H}_3| = \begin{vmatrix} 0 & g_1 & g_2 & g_3 \\ g_1 & L_{11} & L_{12} & L_{13} \\ g_2 & L_{21} & L_{22} & L_{23} \\ g_3 & L_{31} & L_{32} & L_{33} \end{vmatrix}$

Solution : Find the extreme values of the following functions subject to the given constraints.

1) $y = 2x_1 + 2x_1x_2 + x_2$ subject to $3x_1 + 2x_2 = 12$
2) $y = 2x_1^2 + 2x_1x_2 + 3x_2^2$ subject to $3x_1 + 5x_2 = 9$

Solution : Our given objective function is

$$y = 2x_1 + 2x_1x_2 + x_2 \quad \text{subject to } 3x_1 + 2x_2 = 12$$

The lagrange function is defined as

$$L = 2x_1 + 2x_1x_2 + x_2 + \lambda(12 - 3x_1 - 2x_2) \text{-----------}(1)$$

For optimisation, the first order condition requires that

$$\frac{\partial L}{\partial x_1} = 0 \Rightarrow 2 + 2x_2 - 3\lambda = 0 \Rightarrow \lambda = \frac{2 + 2x_2}{3} \qquad (2)$$

$$\frac{\partial L}{\partial x_2} = 0 \Rightarrow 2x_1 + 1 - 2\lambda = 0 \Rightarrow \lambda = \frac{2x_1 + 1}{2} \qquad (3)$$

$$\frac{\partial L}{\partial \lambda} = 0 \Rightarrow 12 - 3x_1 - 2x_2 = 0 \qquad (4)$$

From (2) & (3) $\lambda = \frac{2 + 2x_2}{3} = \frac{1 + 2x_1}{2}$

$$\Rightarrow 3 + 6x_1 = 4 + 4x_2$$

$$\Rightarrow 6x_1 - 4x_2 = 1$$

$$\therefore x_1 = \frac{1 + 4x_2}{6} \qquad (5)$$

Substituting (5) in (4)

$$3 \times \frac{1 + 4x_2}{6} + 2x_2 = 12 \Rightarrow x_2 = \frac{69}{24} \qquad \therefore x_2 = \frac{23}{8}$$

$$\therefore x_1 = \frac{1 + 4x_2}{6} = \frac{1 + 4\left(\frac{23}{8}\right)}{6} = \frac{25}{2}$$

To test the second order condition, let us find the value of $|\overline{H}_2|$

$$|\overline{H}_2| = \begin{vmatrix} 0 & g_2 & g_2 \\ g_1 & L_{11} & L_{12} \\ g_2 & L_{21} & L_{22} \end{vmatrix} = \begin{vmatrix} 0 & 3 & 2 \\ 3 & 0 & 2 \\ 2 & 2 & 0 \end{vmatrix}$$

$$= -3(0 - 4) + 2(6 - 0) = 24 > 0$$

$\therefore$ At $x_1 = \frac{25}{2}$, $x_2 = \frac{23}{8}$, the value of the function is maximum

2. Objective function is $y = 2x_1^2 + 2x_2x_2 - 3x_2^2$

subject to $3x_1 + 5x_2 = 9$

Lagrange multiplier function is defined as

$$L = 2x_1^2 + 2x_1x_2 - 3x_2^2 + \lambda(9 - 3x_1 - 5x_2)$$

The first order condition for optimisation requires

$$\frac{\partial L}{\partial x_1} = 0, \quad \frac{\partial L}{\partial x_2} = 0, \quad \frac{\partial L}{\partial \lambda} = 0$$

$$L_1 = 4x_1 + 2x_2 - 3\lambda = 0 \Rightarrow \lambda = \frac{4x_1 + 2x_2}{3} \qquad (1)$$

$$L_2 = 2x_1 - 6x_2 - 5\lambda = 0 \Rightarrow \lambda = \frac{2x_1 - 6x_2}{5} \qquad (2)$$

$$\frac{\partial L}{\partial \lambda} = 9 - 3x_1 - 5x_2 = 0 \qquad (3)$$

$$(1)\ \&\ (2) \Rightarrow \lambda = \frac{4x_1 + 2x_2}{3} = \frac{2x_1 - 6x_2}{5}$$

$$\Rightarrow 20x_1 + 10x_2 = 6x_1 - 18x_2 \Rightarrow 14x_1 + 28x_2 = 0$$

$$\therefore\ x_1 = \frac{-28x_2}{14} = -2x_2 \qquad \text{-----}(4)$$

Substituting (4) in (3)

$$3 \times (-2x_2) + 5x_2 = 9 \Rightarrow x_2 = -9 \qquad \therefore x_1 = 18$$

Here $g_1 = 3,\ g_2 = 5,\ L_{11} = 4,\ L_{12} = 2,\ L_{21} = 2,\ L_{22} = -6$

$$\therefore\ \left|\overline{H}_2\right| = \begin{vmatrix} 0 & g_1 & g_2 \\ g_1 & L_{11} & L_{12} \\ g_2 & L_{21} & L_{22} \end{vmatrix} = \begin{vmatrix} 0 & 3 & 5 \\ 3 & 4 & 2 \\ 5 & 2 & -6 \end{vmatrix}$$

$$= -3(-8-10) + 5(6-20) = 84 - 70 = 14 > 0$$

$\therefore$ The value of the function is maximum at $x_1 = 18,\ x_2 = -9$

Activity 1

a. Is the constraint always necessary in a real life situation. Think.

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b. Does introduction of Lagrange multiplier change the utility function ? Explain.

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c. Distinguish between Hessian & Bordued Hessian determinant.

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d. For a 5 variables case what should be the condition of optimisation. Do the exercise with pen and paper.

1.4. Economic application of constrained optimisation Utility maximisation & consumer's behavior

Let us take the case of a hypothetical consumer who has to choose between two commodities x & y. For simplicity we will also assume that our hypothetical consumer has positive marginal utility functions. The problem posed here is the maximisation of utility with a given purchasing power B which indicates his budget.

Now we have to maximise his utility function

$$U = U(x, y) \quad (U_x, U_y > 0) \qquad (1)$$

Subject to $$xP_x + yP_y = B \qquad (2)$$

where P_x & P_y are the prices of x & y.

Now it is a problem of utility maximisation. Subject to the budget constraint. Let us formulate a Lagrange function as follows.

$$L = U(x, y) + \lambda(\beta - xP_x - yP_y) \qquad (3)$$

The first order condition of maximisation requires that

$$\frac{\partial L}{\partial x} = L_x = 0, \Rightarrow U_x - \lambda P_x = 0 \therefore = \frac{U_x}{P_x} \text{----------} (4)$$

$$\frac{\partial L}{\partial y} = L_y = 0, \Rightarrow U_y - \lambda P_y = 0 \therefore = \frac{U_y}{P_y} \text{----------} (5)$$

$$\& \quad \frac{\partial L}{\partial \lambda} = L_\lambda = 0 \Rightarrow \beta - xP_x - yP_y = 0 \text{ -------} \quad (6)$$

From (4) & (5) $\quad \lambda = \frac{U_x}{P_x} = \frac{U_y}{P_y}$ ------------------- (7)

$$\Rightarrow \frac{MU_x}{P_x} = \frac{MU_y}{P_y} = \lambda \text{ (Constant)}$$

The first order condition of utility maximisation requires that the ratio of marginal utilities of x & y (U_x & U_y) to this prices should be equal. This is Marshallian condition of Consumer's equilibrium

Equation (7) can be rearranged as

$$\frac{-U_x}{U_y} = \frac{-P_x}{P_y}$$

The first order condition can be given alternative interpretation in terms of indifference curve.

$$U = U(x, y) \Rightarrow dU = U_x d_x + Uy\,dy$$

But along an indifference curve there is no change in utility $\therefore$ du=0

$$\therefore U_x\,dx + U_y\,dy = 0 \Rightarrow \frac{dy}{dx} = -\frac{U_x}{U_y}$$

This represents the slope of indifference curve.

Given budget line is

$$xP_x + yP_y = B$$

$$\therefore\ y = \frac{B}{P_y} - \frac{P_x}{P_y}x$$

$\therefore$ Slope of the budget line is $-\frac{P_x}{P_y}$

In equilibrium, Slope of the budget line = Slope of the indifference curve

$$\Rightarrow -\frac{u_x}{u_y} = -\frac{P_x}{P_y}$$

The second order condition for maximisation is $|\overline{H}_2| > 0$

$$|\overline{H}_2| = \begin{vmatrix} 0 & g_x & g_y \\ g_x & L_{xx} & L_{xy} \\ g_y & L_{yx} & L_{yy} \end{vmatrix}$$

$$= \begin{vmatrix} 0 & P_x & P_y \\ P_x & U_{xx} & U_{xy} \\ P_y & U_{yx} & U_{yy} \end{vmatrix} \qquad \begin{bmatrix} \text{Since } g_x = P_x,\ g_y = P_y \\ L_{xx} = U_{xx}, L_{xy} = U_{xy}, L_{yy} = U_{yy} \end{bmatrix}$$

$$\therefore\ |\overline{H}_2| = 2P_xP_yU_{xy} - P_y^2U_{xx} - P_x^2U_{yy} > 0 \text{ ----------(8)}$$

If utility maximisation requires $|\overline{H}_2|$ to be strictly +ve, we can show that a +ve $|\overline{H}_2|$ means strict convexity of indifference curve at equilibrium. Just as the downward slope of indifference curve is guaranteed by a $\frac{-dy}{dx}\left(=\frac{U_x}{U_y}\right)$, its strict convenity would be ensured by a +ve $\frac{d^2y}{dx^2}$.

Let us derive the expression $\frac{d^2y}{dx^2}$,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(-\frac{U_x}{U_y}\right) = \frac{-1}{U_y^2}\left[U_y\frac{dU_x}{dx} - U_x\frac{dU_y}{dx}\right] \text{ ----------(9)}$$

We know that $U = U(x, y)$

$\Rightarrow U_x = U_x(x,y) \Rightarrow du_x = U_{xx}dx + U_{yx}dy$

$\Rightarrow \dfrac{dU_x}{dx} = U_{xx} + U_{yx}\dfrac{dy}{dx}$ [Dividing both sides by dx]

$\Rightarrow \dfrac{dU_x}{dx} = U_{xx} - U_{yx}\dfrac{P_x}{P_y}$ $\left[\because \dfrac{dy}{dx} = -\dfrac{P_x}{P_y} \text{ at equilibrium}\right]$

Similarly $\dfrac{dU_y}{dx} = U_{xy} - U_{yy}\dfrac{P_x}{P_y}$

Substituting the values of $\dfrac{dU_x}{dx}, \dfrac{dU_y}{dx}$ in (9)

$$\frac{d^2y}{dx^2} = -\frac{1}{U_y^2}\left[U_y\left\{U_{xy} - U_{yx}\frac{P_x}{P_y}\right\} - U_y\frac{P_x}{P_y}\left\{U_{xy} - U_{yy}\frac{P_x}{P_y}\right\}\right]$$

$$= -\frac{1}{U_y^2}\left[\frac{P_y^2U_{xx} - P_xP_yU_{yx} - P_xP_yU_{xy} + P_x^2U_{yy}}{P_y^2}\right]$$

$$= \frac{2P_xP_yU_{xy} - P_y^2U_{xy}}{U_yP_y^2} = \frac{|\overline{H}_2|}{U_yP_y^2} \quad \text{[using ----- 8]}$$

Since $|\overline{H}_2| > 0, U_y > 0, P_y > 0$ $\quad \therefore \dfrac{d^2y}{dx^2} > 0$

$\therefore$ At equilibrium indifference curve is convex to the origin

Illustration :

Given the utility function $U = x^{\frac{1}{3}}y^{\frac{1}{2}}$ and prices of x & y are Rs. 2.& Rs. 6 unit. The income of the consumer is Rs. 40. Find maximises his utility.

Solution : Our objective function is $U = x^{\frac{1}{3}}y^{\frac{1}{2}}$

subject to constraint 2x+6y=40.

The Lagrange function will be

$$L = x^{\frac{1}{3}}y^{\frac{1}{2}} + \lambda(40 - 2x - 6y) \text{---------------} (1)$$

The first order condition for maximisation

$$\frac{\partial L}{\partial x} = 0; \quad \frac{\partial L}{\partial x} = 0; \quad \frac{\partial L}{\partial \lambda} = 0$$

$$\frac{\partial L}{\partial x} = \frac{1}{3}x^{\frac{-2}{3}}y^{\frac{1}{2}} - 2\lambda = 0 \Rightarrow \lambda = \frac{1}{6}x^{-\frac{2}{3}}y^{\frac{1}{2}} \text{-------------} (2)$$

$$\frac{\partial L}{\partial y} = \frac{1}{2}x^{\frac{1}{3}}y^{-\frac{1}{2}} - 6\lambda = 0 \Rightarrow \lambda = \frac{1}{12}x^{\frac{1}{3}}y^{-\frac{1}{2}} \text{------------} (3)$$

$$\frac{\partial L}{\partial \lambda} = 40 - 2x - 6y = 0 \Rightarrow 2x + 6y = 40 \text{-------------} (4)$$

$$(2) \;\&\; (3) \;\Rightarrow = \frac{1}{6}x^{-\frac{2}{3}}y^{\frac{1}{2}} = \frac{1}{12}x^{\frac{1}{3}}y^{-\frac{1}{2}}$$

$$\Rightarrow 2x^{-\frac{2}{3}}y^{\frac{1}{2}} = x^{\frac{1}{3}}y^{-\frac{1}{2}} \Rightarrow 2.\; x^{-\frac{2}{3}-\frac{1}{3}} = y^{-\frac{1}{2}-\frac{1}{2}}$$

$$\Rightarrow 2.\; x^{\frac{-3}{3}} = y^{-\frac{1}{2}-\frac{1}{2}} \Rightarrow 2\;\frac{1}{x} = \frac{1}{y} \Rightarrow x = 2y$$

Substituting the value of x=2y in (4)

$$2y + 6y = 40 \Rightarrow y = 4 \qquad \therefore x = 8$$

The second order condition requires that

$$|\overline{H}_2| = \begin{vmatrix} 0 & g_x & g_y \\ g_x & L_{xx} & L_{xy} \\ g_y & L_{yx} & L_{yy} \end{vmatrix}. \qquad \text{Here } g_x = 2, \quad g_y = 6,$$

$$L_{xx} = \frac{1}{3}\left(-\frac{2}{3}\right)x^{-\frac{1}{3}-1}y^{\frac{1}{3}} = \frac{-2}{9}8^{-\frac{5}{3}}4^{\frac{1}{3}}$$

$$= -\frac{2}{9}2^{3\times\left(\frac{-5}{3}\right)}2^{\frac{2}{3}} = -\frac{2}{9}2^{\frac{-5+2}{3}} = \frac{-1}{9}$$

$$L_{yy} = -\frac{1}{2}x^{\frac{1}{3}}\left(-\frac{1}{2}\right)y^{-\frac{1}{2}-1} = \frac{1}{4}8^{\frac{1}{3}}4^{-\frac{3}{2}} = \frac{1}{4}\;\frac{1}{4} = \frac{1}{16}$$

$$L_{yx} = L_{yx} = \frac{1}{3}x^{\frac{-2}{3}}y^{-\frac{1}{2}} = \frac{1}{3}8^{-\frac{2}{3}}4^{-\frac{1}{2}} = \frac{1}{3}2^{-2}2^{-1} = \frac{1}{3}2^{-3} = \frac{1}{24}$$

$$|\overline{H}_2| = \begin{vmatrix} 0 & 2 & 6 \\ 2 & -\frac{1}{9} & \frac{1}{24} \\ 6 & \frac{1}{24} & \frac{1}{16} \end{vmatrix} = -2\left(\frac{2}{16} - \frac{6}{24}\right) + 6\left(\frac{2}{24} + \frac{6}{9}\right)$$

$$= -2\left(\frac{1}{8} - \frac{1}{4}\right) + 6\left(\frac{1}{12} + \frac{2}{3}\right)$$

$$= \frac{2}{8} + \frac{54}{12} = + \text{ ve quantity}$$

Same $|\overline{H}_2| > 0$, the function has maximum value at y=4 ; x=8

Activity 2

a. Why is there is a constraint in case of utility maximisation? Can a lay man maximise utility without a constraint ? Is it possible ?

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b. Mention two more constrained maximisation problem.

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c. What is objective function ?

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d. Whati is a constraint ?

1.5 Minimisation of Cost

Least Cost Combination of inputs

The technique of constrained optimisation can also be applied in finding least cost input combination (is the quantity of inputs in combination that minimises cost.) for the prediction of a specified level of output Q

Let us assume a smooth production function with two variables L(labour) & K(capital)

$$Q_o = Q(L, K) \text{-------------} (1) \quad (Q_a, Q_b > 0)$$

Now the problem may be formulated as minimisation of cost. If prices of L & K are P_L & P_K , the cost function will be

$$C = P_L L + P_K K \text{-----------} (2)$$

This is our objective function (2) and (1) is our constraint.

Our desired Lagrange function will be

$$Z = P_L L + P_K K + \lambda\left[Q_0 - Q(L, K)\right]$$

The first order condition for minimisation requires

$$Z_L = \frac{\partial Z}{\partial L} = 0 \quad Z_K = \frac{\partial Z}{\partial K} = 0 \text{ and } Z_\lambda = \frac{\partial Z}{\partial \lambda} = 0$$

Now, $Z_L = P_L - \lambda Q_L = 0 \Rightarrow \lambda = \frac{P_L}{Q_L}$ ---------(3)

$$Z_K = P_K - \lambda Q_K = 0 \Rightarrow \lambda = \frac{P_K}{Q_K} \text{--------(4)}$$

$$Z_\lambda = Q_\circ - Q(L,K) = 0 \text{---------------(5)}$$

From (3) & (4) $\lambda = \frac{P_L}{Q_L} = \frac{P_K}{Q_K}$ -----------------(6)

At the point of equilibrium a producer's input price ratio = ratio of Marginal Product of the inputs.

from (6) $-\frac{P_L}{P_K} = -\frac{Q_L}{\partial_K}$-------------(7)

Now, $Q_\circ = Q(L, K) \Rightarrow dQ_\circ = Q_L dL + Q_K dK$

$$\Rightarrow 0 = Q_L \frac{dL}{dK} + Q_K$$

$\Rightarrow \frac{dL}{dK} = -\frac{Q_K}{Q_L}$ is the slope of the isoquant & Slope of the cost curve

$$L = \frac{C}{P_K} - \frac{P_K}{P_L} K$$

Optimal input combination requires that Slope of isoquant

= Slope of cost curve

$$\Rightarrow \frac{-Q_K}{Q_L} = -\frac{P_K}{P_L}$$

The second order condition requires that for minimisation $|\overline{H}_2| < 0$

$$\therefore |\overline{H}_2| = \begin{vmatrix} 0 & g_L & g_K \\ g_L & Z_U & Z_{IK} \\ g_K & Z_{KL} & Z_{KK} \end{vmatrix} = \begin{vmatrix} 0 & Q_L & Q_K \\ Q_L & -\lambda Q_{LL} & \lambda Q_{LK} \\ Q_K & -\lambda Q_{KL} & -\lambda Q_{KK} \end{vmatrix} < 0$$

$$\Rightarrow \lambda\{Q_{KK}Q_L^2 + Q_{IK}Q_K^2 - 2Q_L Q_{LK}\} \langle 0 \qquad [\because Q_{LK} = Q_{KL}]$$

The second order condition for maximisation requires that the isoquanf should be convex at the point of equilibrium. Convenity of isoquant requires that

$$\frac{d^2L}{dK^2} > 0$$

Now, $\frac{d^2L}{dK^2} = \frac{d}{dK}\left(\frac{dL}{dK}\right) = \frac{d}{dK}\left(-\frac{Q_K}{Q_L}\right)$

$$= \left[\frac{Q_L \frac{d}{dK} Q_K - Q_K \frac{d}{dk} Q_L}{Q_L^2}\right] \text{------(8)}$$

[applying quotient rule]

Similarly we know $Q_L = Q_L(L, K)$

$$\Rightarrow dQ_L = Q_{LL} d_L + Q_{LK} dk$$

$$\Rightarrow \frac{d}{dK} Q_L = Q_{LK} + Q_{LL} \frac{dL}{dK}$$

$$= Q_{LK} - Q_{LL}\left(\frac{Q_K}{Q_L}\right)$$

Similarly $\frac{dQ_K}{dK} = Q_{KK} - Q_{KL} \frac{Q_K}{Q_L}$

Substituting these two values $\frac{dQ_K}{dK}$ & $\frac{dQ_L}{dK}$ in (8)

$$\frac{d^2L}{dK^2} = -\left[\frac{Q_L\left\{Q_{KK} - Q_{KL}\left(\frac{QK}{QL}\right) - Q_K\left\{Q_{KL} - Q_{LL}\frac{Q_K}{Q_L}\right\}\right.}{Q_L^2}\right]$$

$$= -\frac{1}{Q_L^3}\left\{Q_{KK} Q_L^2 - 2Q_K Q_L Q_{KL} + Q_{LL} Q_K^2\right\}$$

Since the quantity within parentheses is a -ve quantity according to the condition of maximisation.

$\frac{d^2L}{dK^2} > 0$ $\therefore$ The isoquant is convex to the origin at the point of equilibrium

Illustration : Minimise the cost function

$$C = 2L + 4K \quad \text{subject to } Q = 8L^{\frac{1}{4}}K^{\frac{1}{2}} = 64$$

Solution : Here we have to minimise cost. So our objective function is cost function

$$C = 2L + 4K \quad \text{Subject to } Q = 8L^{\frac{1}{4}}K^{\frac{1}{2}} = 64$$

$$Z = 2L + 4K + \lambda\left(64 - 8L^{\frac{1}{4}}K^{\frac{1}{2}}\right)$$

$$Z_L = \frac{\partial Z}{\partial L} = -2 - \lambda 8 \frac{1}{4} L^{\frac{1}{4}-1} K^{\frac{1}{2}} = 0 \Rightarrow \lambda = \frac{1}{L^{-\frac{3}{4}}K^{\frac{1}{2}}} \text{-------(1)}$$

$$Z_K = \frac{\partial Z}{\partial K} = 4 - \lambda 8 \frac{1}{2} L^{\frac{1}{4}} K^{\frac{1}{2}-1} = 0 \Rightarrow \lambda = \frac{1}{L^{\frac{1}{4}}K^{-\frac{1}{2}}} \text{------(2)}$$

$$Z_\lambda = 64 - 8L^{\frac{1}{4}}K^{\frac{1}{2}} = 0 \text{-------------------(3)}$$

From (1) & (2) $\lambda = \frac{1}{L^{-\frac{3}{4}}K^{\frac{1}{2}}} = \frac{1}{L^{\frac{1}{4}}K^{-\frac{1}{2}}} \Rightarrow \frac{L^{\frac{1}{4}}K^{-\frac{1}{2}}}{L^{-\frac{3}{4}}K^{\frac{1}{2}}} = 1 \Rightarrow \frac{L}{K} = 1$

$$\Rightarrow L = K \text{--------(4)}$$

Using (4) in (3) $8L^{\frac{1}{4}}L^{\frac{1}{2}} = 64 \Rightarrow L^{\frac{3}{4}} = 8 \Rightarrow L = \left(2^3\right)^{\frac{4}{3}} = 2^4 = 16$

$$\therefore K = 16$$

$$\therefore \lambda = \frac{1}{16^{\frac{1}{4}}16^{-\frac{1}{2}}} = \frac{1}{2\,2^{-2}} = \frac{1}{2^{-1}} = 2$$

Let us establish second order condition for minimisation $|\overline{H}_2| < 0$

$$\text{Here } |\overline{H}_2| = \begin{vmatrix} 0 & g_L & g_K \\ g_L & Z_{LL} & Z_{LK} \\ g_K & Z_{KL} & Z_{KK} \end{vmatrix}$$

$$g_L = -8\frac{1}{4}L^{\frac{1}{4}-1}K^{\frac{1}{2}} = (-8)\frac{1}{4}16^{\frac{-3}{4}}16^{\frac{1}{2}} = -1$$

$$g_K = (-8)\frac{1}{2}L^{\frac{1}{4}}K^{\frac{1}{2}-1} = (-8)\frac{1}{2}16^{\frac{1}{4}}16^{-\frac{1}{2}} = -2$$

$$L_{LL} = (-\lambda)2\left(-\frac{3}{4}\right)L^{-\frac{3}{4}-1}K^{\frac{1}{2}} = (-2)\,2\left(-\frac{3}{4}\right)(16)^{\frac{-7}{4}}16^{\frac{1}{2}}$$

$$= 3\,2^{-7}\,2^{2} = 3\,2^{-5} = \frac{3}{32}$$

$$Z_{LK} = (-2)\lambda\frac{1}{2}L^{-\frac{3}{4}}K^{\frac{1}{2}-1} = (-2)\,2\,\frac{1}{2}16^{-\frac{3}{4}}16^{\frac{-1}{2}}$$

$$= (-2)\;2^{-3}\,2^{-2} = -2^{1-3-2} = \frac{-1}{16}$$

$$Z_{KK} = \lambda\,4(-\frac{1}{2})L^{\frac{1}{4}}K^{\frac{-1}{2}-1} = (12)\,2\left(2^4\right)^{\frac{1}{4}}\left(2^4\right)^{-\frac{3}{2}}$$

$$= (+2)\,2\,2\,2^{-\frac{6}{2}} = +2^{1+1+1-3} = 1$$

Activity 3

a. Give the economic interpretation of first order condition for cost minimisation.

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b. Do we arrive at the same condition which are applicable in economic theory by applying mathematical cost minimisation technique?

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c. Compare the first order Condition for profit maximisation & cost minimisation.

1.6 Summary :

In unit 1, we have gone through a new technique of optimisation with constraint. In a real life situation we are given certain conditions. We can't go beyond it. So, mathematics has given a suitable inter pretation of such problems with the use of Lagrange function.

Such optimisation technique can locate relative maxima & minima & absolute optimum determination is beyond the scope of this chapter.

Everything is done in this chapter with the help of calculas & the severe most limitation of this technique is its inability to deal with constraints with inequality form. For such problems, We use techniques called linear programming. The remaining portion of this book will explain it.

1.7 Further readings :

Allen R. G. D. : Mathematical Analysis for economists (1938) Machillan & Co.

Mehta B. C. and G. M. K. Madrani (1997) : Mathematics for economists S. Chand & Sons.

1.8 Exercise for Self Assessment

1. Given the utility function $K = x^{\frac{1}{3}} y^{\frac{1}{2}}$ subject to the budget constraint $2x + 6y = 40$. Find the combination of the purchase of x & y. Which will maximise utility of the consumer.

2. A producer produces 72 units output with the cost function $C = 0.5L + 9k$ subject to $Q = 3L^{\frac{1}{2}} K = 72$. Find the combination of inputs which will maximise cost.

3. Show that diminishing marginal utility doesn't imply the convexity of indifference curve does convexity imply diminishing marginal utility.

[**Hints :** Convexity implies $\frac{d^2y}{dx^2} > 0$ & prove this condition for the indiffer curve $K = K(x, y)$].

4. If $K = x^2y$ is a utility function. Find the combination of x & y which leads the consumer to equilibrium.

●●●

UNIT-2

CALCULAS FOR DYNAMIC ANALYSIS

2.1 : Introduction

2.2 : Objectives

2.3 : First Order Differential Equation and its Solutions:

2.3 (i) Application to dynamic stability of market and

2.3 (ii) Simple growth process (Harrod-Domar)

2.4 : First Order Difference Equation and its Solutions

2.4 (i) Application of difference equation lagged marker model (cobweb)

2.4 (ii) Harrods' model of Growth

2.5 : Summery

2.6 : Additional Reading

2.7 : Self Assessment Test

2.1: Introduction:

For dynamic analysis in the economy, calculas is used. The theory of differential equations has become an essential tool of economic analysis. It would be difficult to comprehend the contemporary literature of economics if one does not understand basic concepts and results of modern theory of differential equations.

A differential equation expresses the rate of change of the current state as a function of the current state. The unit mainly covers the first order differential equation and its solutions. Its application is done in the dynamic stability of the market and simple growth process. The differential equation is used in the Harrod-Domar growth model.

Moreover, in the application of lagged market model and Harrods growth model, the first order difference equation is used. A difference equation measures the change in the value of y into successive periods of time. A general first order difference equation takes the form–

$$x_t = f(t, x_{t-1}) \text{ for all } t.$$

In particular, given any value x_0, there exists a unique solution path, x_1, x_2, ...

2.2 : Objectives :

On reading this unit, you should be in a position to—

- Understand the concepts and differences of differential equation and difference equation.
- Become efficient in applying the equations in the dynamic analysis of market model.

2.3 : First Order Differential Equation and Its Solutions :

a) With constant Coefficient and constant term :

In a differential equation when the derivative $\frac{dy}{dx}$ and the dependent variable y appear in the first degree and no product of the form $y\left(\frac{dy}{dx}\right)$ appears, then the equation is said to be linear. So, in general form, the first order differential equation will be —

$$\frac{dy}{dx} + u(x)y = v(x) \quad \text{..........} \quad (1)$$

Where u and v like y, are two functions of x. Here, if u and v are constants, then the first order linear diffrential equation reduces to—

$$\frac{dy}{dx} + ay = b \quad \text{................} \quad (2)$$

This is called first order linear differential equation with constant coefficient and constant term. This is said to be <u>homogonous</u>. If the constant term b = 0, equation (2) can be written as–

$$\frac{dy}{dx} + ay = 0 \quad \text{...............} \quad (3)$$

This form is said to be homogenous. This can be rewritten as

$$\frac{1}{y}\frac{dy}{dx} = -a \quad \text{..................} \quad (4)$$

$$\Rightarrow \log y = \int -adx$$

Since, $\log\ y = x$

$$\Rightarrow y = e^x$$

$$\Rightarrow y = e^{\int -adx} \Rightarrow y = e^{ax+c} \Rightarrow y = e^{-ax}e^c \Rightarrow y = Ae^{-ax} \quad (5)$$

Putting $x = 0$, $y(0) = A$ and so definite solution is

$$y(x) = y(0)Ae^{-ax} \quad (6)$$

But in case of non homognous first-order diffrential equation when $b \neq 0$ will be—

$$\frac{dy}{dx} + ay = b$$

Here, the soultion will consist of two parts— complementary solution and particular solution. The sum of these two parts will give the complete generqal solution.

The complementary solution is denoted as "y_c" is the solution of the homognous form i.e. $\frac{dy}{dx} + ay = 0$ when $y_c = Ae^{-ax}$. In case particular integral, suppose y=c (constant)

Therefore, $\frac{dy}{dx} = 0$. The equation can bewritten as $ay = b$

$$\text{or } y(= y_p) = \frac{b}{a}, a \neq o$$

Therefore, the complete solution is given by the sum of y_c and y_p.

Thus, the general solution of complete equation will be—

$$y(x) = y_c + y_p$$

$$= Ae^{-ax} + \frac{b}{a} \quad \text{.................(7)} \quad \text{where } a \neq 0$$

If $x = 0$, $\quad y(0) = Ae^0 + \frac{b}{a} \quad \therefore A = \left[y(0) - \frac{b}{a}\right]$.......... (8)

Substituting (8) in (7),

$$y(x) = \left[y(0) - \frac{b}{a}\right]e^{-ax} + \frac{b}{a} \quad \text{........... (9)}$$

Such that $a \neq 0$.

Numerical Examples: Solve the equation $\frac{dy}{dx}+6y=12$ with the initial condition y (0) = 5

Sol: The complementary function is

$$\frac{dy}{dx}+6y=0 \qquad \therefore y_c = Ae^{-6x}$$

The particular solution $y_p=\frac{b}{a}=\frac{12}{6}=2$ putting x=0, y(0) = 5 from initial condition.

so the constant A will be $y(0)-\frac{b}{a}=5-2 \;=3$

∴ The final solution is—

$$y(x)=\left[y(0)-\frac{b}{a}\right]e^{-ax}+\frac{b}{a}= .$$

$$\Rightarrow y(x)=3e^{-6x}+2$$

b) First Order Differential Equation with variable coefficient and variable term :

In the differential equation (i) $\left(\frac{dy}{dx}+ay=b\right)$, the a and b are constants.

If these constants are replaced by variables which are functions of x, then it becomes.

$$\frac{dy}{dx}+u(x)y=v(x) \qquad (1)$$

Here, u(x) and v(x) are two functions of x. To solve the equation, suppose equation (i) is homogonous, where $v(x)=0$

The differential equation becomes

$$\frac{dy}{dx}+u(x)y=o \Rightarrow \frac{1}{y}\frac{dy}{dx}=-u(x) \qquad \text{..........} \; (2)$$

If both sides of equation is integrated with respect to x such that—

$$\int\frac{1}{y}dy=-\int u(x)dx \quad \text{then}$$

$$\int \frac{1}{y} dy = -\int u(x)dx \Rightarrow \log y = -c - \int u(x)dx \Rightarrow y(x) = e^{\log y}$$

$$\Rightarrow y(x) = e^{-c-\int u(x)dx}$$

$$\Rightarrow y(x) = Ae^{-\int u(x)dx} \qquad (3)$$

where $A = e^{-c}$

Equation (3) gives the general solution of the differential equation of homogenous form with variable coefficient given by equation (4). In equation (3), the integral $\int u(x)dx$ can be obtained only if $u(x)$ function is explicitly defined.

For example : if the equation is

$$\frac{dy}{dx} + 3x^2y = 0 \qquad (4)$$

Here $u(x) = 3x^2$ and so $u(x)dx = 3\frac{1}{3}x^3 + c_1 = x^3 + c_1$

$$\therefore \int u(x)dx = x^3 + c_1$$

The soultion of equation (4) will be

$$\therefore y(x) = Ae^{-\int u(x)dx} = Ae^{-(x^3+c_1)} \Rightarrow y(x) = Ae^{-c_1}.e^{-x^3} = Be^{-x^3}$$

where, $B = Ae^{-c_1} = e^{-c}.e^{-c_1} = e^{-(c+c_1)}$

As, A and B are arbitacary constants, they can be used synonymously. The general solution is

$$y(x) = Ac^{-x^3}.$$

Therefore in case of non-homogenous case,

$$\frac{dy}{dx} + u(x)y = v(x) \qquad (5)$$

The general solution is given by

$$y(x) = c^{\int udx}\left[A + \int ve^{udx}dx\right]$$

c) Exact Differential Equation:

Exact differential equation may be both linear and non liner. If there is function of two variables F (x,y); its total differential is given by

$$dF(x,y) = \frac{dF}{dx}dx + \frac{dF}{dy}dy$$

When the total differential is made equal to zero, the resulting equation

$$\frac{dF}{dx}dx + \frac{dF}{dy}dy = 0 \qquad (6)$$

is called an Exact differential equation. This is because of the fact that the left side of equation (6) is exactly the differential of the function F(x,y).

In general, an exact differential equation is given by

$$Mdy + Ndx = 0 \quad \ldots\ldots\ldots\ldots\ldots (7)$$

If and only if there exist a function F(x,y) such that $M = \frac{dF}{dy}$ and $N = \frac{dF}{dx}$

2.3 (i) : Application to Dynamic Stability of Market Price :

The solution of differential equation enables us to obtain the conditions for dynamic stability of equilibrium. To demonstrate the dynamic stability of market price, the single commodity market model have been taken into account———

$$Q_d = b - cP \qquad (b, c > 0) \qquad (1)$$

$$Q_s = -d + eP \qquad (d, e > 0) \qquad (2)$$

$$Q_d = Q_s \qquad (3)$$

The equilibrium price is given by

$$\overline{P} = \frac{b+d}{c+e} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

If the initial price P (0) is exactly to the equilibrium price, the market price is stable and there is no need of any dynamic analysis of price. But, if the initial price is different from the equilibrium price, that is if $P(0) \neq \overline{P}$, Then market will be unstable as there will be divergence between demand and supply at a given price. Finally, the price will be stabilized in due course through a process of adjustment overtime. During this time price as well as Q_d and Q_s Will also change.

In dynamic stability, the time path of Pt has to be tested so that the pattern of price change can be determined.

The change in price is governed by relative strength of demand and supply. Therefore, it can be considered that the change in price over time is directly Proportional to excess demand $(Q_d - Q_s)$ such that

$$\frac{dp}{dt} = \alpha(Q_d - Q_s) \qquad (\alpha > o) \qquad (5)$$

Where α represents the adjustment coefficent which remains constant over time. Now, Substituting Q_d and Q_s from (1) and (2) in (5)

$$\frac{dp}{dt} = \alpha[b - cP + d - eP] = \alpha(b+d) - \alpha(c+e)P$$

$$\Rightarrow \frac{dp}{dt} + \alpha(c+e)P = \alpha(b+d) \qquad \text{..................} (6)$$

Equation (6) represents a first order differential equation with constant coefficient and constant term. To get the time path P_t, we have to find out the solution of the differential equation.

Following the general solution of differential equation.

$$\frac{dy}{dx} + ay = b \text{ as } \quad y(x) = \left[y(0) - \frac{b}{a}\right]e^{-ax} + \frac{b}{a}$$

Eq^n (6) be written as

$$p_t = \left[p_o - \frac{\alpha(b+d)}{\alpha(c+c)}\right]c^{-\alpha(c+c)p} + \frac{\alpha(b+d)}{\alpha(c+e)}$$

Or $p_t = [p_o - \bar{p}]e^{-\beta t} + \bar{p}$ .... (7)

where $\bar{p} = \dfrac{b+d}{c+e}$ and $\beta = \alpha(c+e) > 0$. Since α, c and e are positive.

In equation (7), the time path of P_t is obtained. It is seen that regarding whether the initial price P(0) is greater than equilibrium price $\bar{p}$ or less than $\bar{p}$, p_t will tend to $\bar{p}$ when $t \to \alpha$. This is because of the fact that $\beta > 0$ and so $e^{-\beta t} = \dfrac{1}{e^{\beta t}}$ will tend to zero as $t \to \alpha$. Thus finally the time path of P_t will Converge to the level $\bar{p}$ and so the equilibrium is said to be dynamically stable.

Numerical Example:

Let the demand and supply functions be given by

$$Q_d = 8 - 2P, \qquad Q_s = -10 + 20P$$

It is also assumed that the change in price over time is equal to 2 times of excess demand such that.

$$\frac{dp}{dt} = 2(Qd - Qs)$$

To obtain the stability condition of price.

$$\frac{dp}{dt} = 2[8 - 2P + 10 - 20P] \Rightarrow \frac{dp}{dt} + 44P = 46$$

Following, the solution of differential equation

$$P_t = \left[P(0) - \frac{b}{a}\right]e^{-at} + \frac{b}{a} \qquad \text{where } \frac{b}{a} = \frac{46}{44} = \bar{P} = \frac{23}{22}$$

$$\therefore P_t = \left[P(0) - \frac{23}{22}\right]e^{-44t} + \frac{23}{22}$$

2.3 (II): SIMPLE GROWTH PROCESS (HARROD-DOMAR) :

The Harrod-Domar growth theory basically combines the multiplier and accelerator principles to explain the process of steady and progressive growth of income overtime. The Underlying idea of growth theory is that if the autonomous investment increases, the corresponding increase in output will be $\frac{A}{1-C}$, where c is the marginal propensity to consume $(1 > C > O)$ and A is the autonomous investment. The expansion of output makes accelerator operative which leads to increase in induced investment. The additional induced investment will again be multiplied due to the operation of multiplier and another round of increase in output begins. The process continues and output expands progressively. In a national income model, the autonomous expenditure both consumption and expenditure are taken into account.

$$Y = C + I + A \qquad \text{(I)}$$

where C = cy where c marginal propensity to consume.

$$\text{and } I = v\frac{dy}{dt},$$

Where v is accelerator and A is autonomous investment. Therefore, on substitution C and I in (i) we get.

$$y = cy + v\frac{dy}{dt} + A$$

$$\Rightarrow v\frac{dy}{dt} = (1-C)Y - A$$

$$\Rightarrow \frac{dy}{dt} = \frac{(1-C)}{v}y - \frac{A}{v}$$

$$\Rightarrow \frac{dy}{dt} = \frac{s}{v}y - \frac{A}{v}$$

$$\Rightarrow \frac{dy}{dt} = \frac{s}{v}\left(y - \frac{A}{s}\right)$$

where, $s = 1 - c$ equal to marginal Propensity to save

$$\frac{dy}{dt} = \alpha\left(y - \frac{A}{s}\right) \quad \text{where } \alpha = \frac{s}{v}$$

The solution of equation (2) will give the time path of growth in output. Therefore, the solution of the model depends on the assumption of A-Whether the autonomous investment is fixed or it is progressive.

Case I: When A is find.

Let us substitute $y = Y - \frac{A}{s}$

$$\therefore \frac{dy}{dt} = \frac{dY}{dt}$$

and from equation (II), $\frac{dy}{dt} = \alpha y$ (III) where $\alpha = \frac{s}{v}$

The solution of equation (III) is given by

$$y(t) = y(0)e^{\alpha t} \qquad \text{(IV)}$$

The solution of equation (IV) shows that there is progressive growth of income or output over time exponentially at a rate $\alpha = \frac{s}{v} > 0$

Case II : When autonomous investment is progressive; $A = A_0e^{rt}$

When the autonomous investment instead of being fined, increases exponentially at a rate $r, (r > 0)$, Then equation (II) becomes

$$\frac{dy}{dt} = \alpha\left(y - \frac{A_0 e^{rt}}{s}\right) \qquad \text{(IV)}$$

If $y = \bar{y}(0)e^{rt}$ as solution of equation (IV) taking progressive growth of output being equal to progressive growth of investment.

Thus, $\frac{dy}{dt} = r\bar{y}(0)e^{rt}$

$\therefore$ Equation (IV) can be written as

$$r\bar{y}(0)e^{rt} = \alpha\left[\bar{y}(0)e^{rt} - \frac{A_0}{s}e^{rt}\right] \Rightarrow r\bar{y}(0) = \alpha\bar{y}(0) - \frac{\alpha A_0}{s}$$

$$\Rightarrow (\alpha - r)\bar{y}(0) = \frac{\alpha}{s}A_0 = \frac{A_0}{v} \qquad \left[\text{where} \quad \alpha = \frac{s}{v}\right]$$

$$\therefore \bar{y}(0) = \frac{A_0}{v(\alpha - r)}$$

So, the value of $\bar{y}(0)$ is fixed by the structural parameters of the model and constants.

2.4: FIRST ORDER DIFFERENCE EQUATION AND ITS SOLUTION:—

a) FIRST ORDER DIFFERENCE EQUATION

The change in y due to change in time is represented by $\frac{\Delta y}{\Delta t}$ when time variable is discrete. Since the time variable takes integer value, the change in the value of y into successive period is represented by Δy as $\Delta t = 1$ The value of Δy is called "a first finite difference of y". Thus, Δy represents $(y_{t+i} - y_t)$. Where y_t is the value of y in period t and y_{t+i} is the value of y in period $(t+i)$. Similarly, when the change in the value of Δy_t over to successive time period is found out, it is symbolized by $\Delta^2 y$ Which is equal to $(\Delta y_{t+i} - \Delta y_t)$. $\Delta^2 y$ is callẽd second finite difference and in general term $\Delta^2 y$ is called K[th] finite difference.

For example—

$$\Delta y_t = (y_{t+i} - y_t) = 4 \qquad \text{(I)}$$

or $\Delta y_t = (y_{t+i} - y_t) = 0.2y_t$ (II)

Equation (i) can be written as

$$y_{t+i} = y_t + 4 \quad (3)$$

Similarly, equation (II) can be rewritten as

$$y_{t+i} = y_t - 0.2y_t \Rightarrow y_{t+i} = 0.8y_t \quad (4)$$

The equations (3) and (4) are called difference equation. Since these is time log of one period in both equations, they are called "first order difference equations."

b) Solution of First Order Difference Equation :

Incase of differénce equation, the value of y for a given time period t is determined. There are two methods of solving the first order difference equation —

i) Iterative Method of Solution :

In this method, the value of y_t is substituted corresponding to the value of t = 1, 2, 3,... From example—

$y_{t+i} = y_t + 2$. Now putting $t = 0$, $y_1 = y_0 + 2$

Again Putting $t = 1$ in equation (I)

$$y_2 = y_1 + 2 = (y_0 + 2) + 2 \qquad \therefore y_2 = y_o + 2(2)$$

Again Putting $t = 2$, in equation (I)

$$y_3 = y_2 + 2 = \{y_0 + 2(2)\} + 2 = y_0 + 3(2)$$

In general form, $y_t = y_0 + 2t$

Similarly, if $y_{t+i} = 0.5y_t$

when $t = 0$, $y_1 = 0.5y_0$

when $t = 1$, $y_2 = 0.5y_1 = 0.5(0.5y_0) = y_0(0.5)^2$

Similarly $t = 2$, $y_3 = 0.5y_2 = 0.5[0.5(0.5)y_0] = y_0(0.5)^3$

In general form, $y_t = y_0(0.5)^t$.

Now, if there is homogenous first order difference equation of the form–

$$my_{t+1} - ny_t = 0 \quad \text{(i)}$$

$$\Rightarrow y_{t+1} = \frac{n}{m} y_t \quad \text{(ii)}$$

Following interactive process, the solution of (II) is given by

$$y_t = \left(\frac{n}{m}\right)^t y_0 \qquad \text{(iii)}$$

This can be written as $y_t = Ab^t$ where $b = \frac{n}{m}$ and $A = y_0$

GENERAL METHOD OF SOLUTION :

Suppose, There is a first order difference equation of the following form,

$$y_{t+1} + ay_t = c \qquad \text{(I)}$$

where a and c are constants. Like differential equation, the general solution will also consist of two parts Complementary solution and particular solution.

The complementary solution is the general solution of the homogenous part—

$$y_{t+1} + ay_t = 0 \qquad \text{(II)}$$

This can be written as

$$y_{t+1} = Ab^t \qquad \text{(III)} \quad [\text{following } y_t = y_0\left(\tfrac{n}{m}\right)^t \; y_t = Ab^t]$$

Substituting equation (III) in (II) the complementary solution is

$$Ab^{t+1a} + aAb^t = 0 \Rightarrow Ab^t(b+a) = 0 \Rightarrow b+a = 0 \text{ or } b = -a$$

The complementary solution of equation (III) becomes

$$y_c = Ab^t = A(-a)^t$$

For attaining particular solution, suppose $y_t = k$ (constant) and so $y_{t+1} = k$ Then equation (i) becomes.

$$k + ak = c \Rightarrow k(1+a) = c \quad \therefore k = \frac{c}{1+a}$$

$\therefore$ The particular solution in this case is

$$y_p = k = \frac{c}{1+a} \qquad \text{(iv)}$$

provided that $a \neq -1$.

But if $a = -1$, y_p is not defined. In that case, the particular solution is considered as –

$y_t = kt$ and so $y_{t+1} = k(t+1)$.

Now, from equation (I)

$$k(t+1)+akt=c \Rightarrow k(1+t+at)=c \Rightarrow k=\frac{c}{(1+t+at)}$$

$$\therefore y_p = kt = \frac{ct}{(1+t+at)}$$

$\therefore$ The complete solution will be —

$$y_t = A(-a)^t + \frac{c}{1+a} (a \neq -1) \qquad \text{(V)}$$

and $y_t = A(-a)^t + \dfrac{ct}{1+t+at} (a \neq -1)$

or $\quad y_t = A + ct \qquad$ (since $a = -1$) $\qquad$ (VI)

If $t = 0$, we will be able to determine the arbitary constant A. By putting $t = 0$, in (V) and (VI).

$$y_0 = A + \frac{c}{1+a}$$

$$\therefore \quad A = \left(y_0 - \frac{c}{1+a}\right) \qquad \text{(VII)}$$

From (VI)

$$y_0 = A + 0 \quad \text{or} \quad y_0 = A$$

The final Solution is ——

$$y_t = \left(y_0 - \frac{c}{1+a}\right)(-a)^t + \frac{c}{1+a} \qquad (a \neq -1)$$

and $\qquad y_t = y_o + ct \qquad (a = -1)$

2.4 (I) Application of Difference Equation lagged market model (cobweb) :

In a market situation where the output decision of producer depends on the price of the previous period and the demand for the product being function of current period price, this market model is popularly known as Cobweb model. This market model is very appropriate for agricultural product or for the product of perishable commodities.

Suppose, a market model is–

$$Q_{dt} = f(p_t) \quad \text{and} \quad Q_{dt} = g(p_{t-1})$$

A linear version of lagged supply and unlagged demand function is considered and a simple market model of single commodity is

$$Q_{dt} = a - bp_t, \qquad a, b > 0 \qquad (1)$$

$$Q_{st} = -c + dp_{t-1}, \qquad c, d > 0 \qquad (2)$$

$$Q_{dt} = Q_{st} \qquad (3)$$

Now substituting (1) and (2) in equation (3), we get.

$$a - bp_t = -c + dp_{t-1} \Rightarrow bp_t + dp_{t-1} = a + c$$

$$\Rightarrow p_t + \frac{d}{b} p_{t-1} = \frac{a+c}{b} \qquad (4)$$

Shifting the time subscript ahead of one period, from eqn (4)

$$p_{t+1} + \frac{d}{b} p_t = \frac{a+c}{b} \qquad (5)$$

Equation (5) is a first order difference equation.

Comparing equation (5) with the general first order difference equation we get.

$$p_t = \left(p_0 - \frac{\frac{a+c}{b}}{1+\frac{d}{b}}\right)\left(-\frac{d}{b}\right)^t + \frac{\frac{a+c}{b}}{1+\frac{d}{b}}$$

$$\Rightarrow p_t = \left(p_o - \frac{a+c}{b+d}\right)\left(-\frac{d}{b}\right)^t + \frac{a+c}{b+d}$$

$$\Rightarrow p_t = (p_o - \bar{p})\left(-\frac{d}{b}\right)^t + \bar{p} \qquad (6)$$

where $\bar{p} = \frac{a+c}{b+d}$ is the inter temporal equilibrium price.

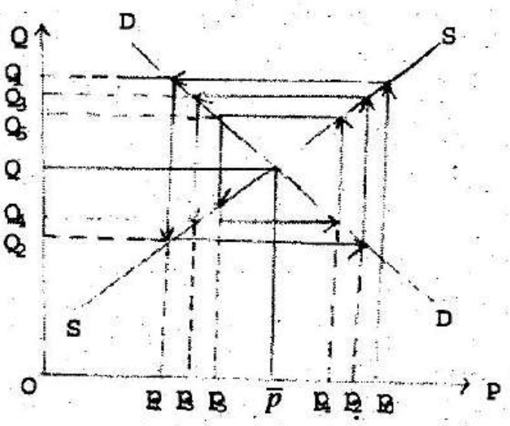
Thus, it is seen that the time path of price of the cobweb market model depends on the ratio $\frac{d}{b}$. Since d is slope of supply curve and b is the slope of demand curve, the time path of price depends on the slopes of demand and supply curve.

There are three situation depending on the relative value of the slopes of demand and supply curves——

Case I :

When the slope of demand curve is greater than the slope of supply curve or when $d > b$, $\frac{d}{b} > 1$. So, when the value of t increases in equation(6), the value of the $\left(\frac{b}{d}\right)^t$ will go on falling. So as "t" approaches to infinity or when t is very large $\left(\frac{d}{b}\right)^t$ Will tend to zero. In that case, p_t will be equal to $\overline{p}$. Thus, when slope of demand curve is greater than the slope of supply curve even if the initial price p_o is different from equilibrium price $\overline{p}$, the current price will approach to equilibrium price when time period is long enough. Hue, the price will be dynamically stable at equilibrium level $\overline{p}$ and time path of price is known as convergent.

This situation is shown in figure 2:1



Fgure :21

In figure 2.1, the equilibrium price $\overline{P}$, equilibrium quantity $\overline{Q}$, are given by the point of intersection of demand and supply curve. In the figure, it is seen that if $b > d$, at the initial price p_0, supply exceeds demand and so price falls to p_1 in period 1. Again, at price p_1, demand exceeds supply. So price increases from p_1 to p_2. The process continues and the gap between equilibrium current price and the equilibrium price goes on decreasing and finally the current price converges to $\overline{p}$ So, the market will be dynamically stable.

Case II :

When the slope of supply curve is steeper than the demand curve or when $b<d$ then $\frac{d}{b} > 1$. So as time increases, $\left(\frac{-d}{b}\right)^t$ will go on increasing and finally $\left(-\frac{d}{b}\right)^t$ will tend to infinity as "t" tends to infinity. In such a situation, the current price will deviate more and more from the equilibrium price. The Market price will never became stable and the time path, P_t will be explosive and popularly called divergent time path. The situation can be presented in fig: 2.2 :

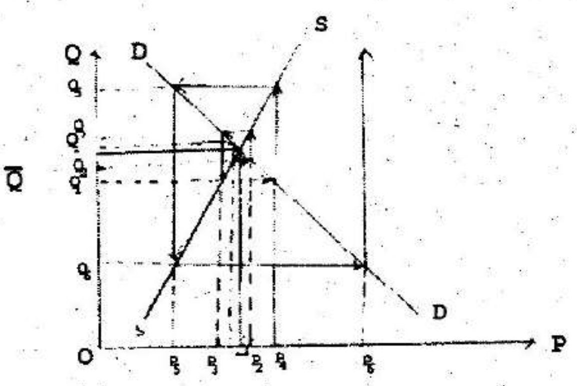


Fig. 2.2

In the figure, it is been that, at initial price P_0 supply exceeds demand and so the price falls to P_1 in time period 1. At price P_1, demand exceeds supply and so price increases to P_2. Again at P_2, supply exceeds demand and so price falls to P_3. This process continues leading to divergence of current price from the equilibrium price.

Case III :

When the slope of both demand and supply curve are the same or when b=d, then $\frac{d}{b} = 1$.

In such a case $\left(-\frac{d}{b}\right)$ Will be alternatively +1 and -1 depending on whether t is even or odd. Therefore, the divergence between the current price and equilibrium price remains the same and the time path is said to be regular. This situation is explained in the figure 2.3. The figure shows the divergence between current price and equilibrium price is remaining the same.

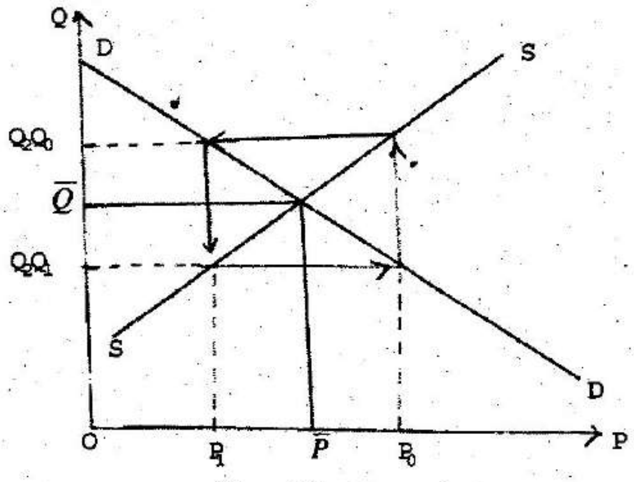


Figure :2.3

The time path of quantity Q_t by substituting the solution of equation (b) in (1), we have,

$$Q_t = a - b\left[\{p_0 - \overline{p}\}\left(-\frac{d}{b}\right)^t + \overline{p}\right]$$

or $$Q_t = -b\left(p_0 - \overline{p}\right)\left(-\frac{d}{b}\right)^t + \left(a - b\overline{p}\right)$$

2.4. (II) Harrod Model of growth :

In Harrod model of growth, it is assumed that

a) Actual saving is a fixed proportion of income such that

$$S_t = \alpha y_t \qquad 1 > \alpha > 0 \qquad (1)$$

where S_t is savings in time "t" and y_t is income in time t.

α is the overage propensity to save.

b) The investment is constant proportion of the difference betwwen current years' output to last years' output such that—

$$I_t = \delta(y_t - y_{t-1}), \qquad \delta > 0 \qquad (2)$$

where I_t is investment in time "t" and δ is acceleator.

c) Macro economic equilibrium requires the equation between planned investment and realised saving such that

$$I_t = S_t \qquad (3)$$

Now, substituting (1) and (2) in (3), we have,

$$\alpha y_t = \delta(y_t - y_{t-1}) \Rightarrow \alpha y_t - \delta y_t = -\delta y_t \Rightarrow y_t = -\frac{\delta}{\alpha-\delta} y_{t-1}$$

$$\Rightarrow y_t + \frac{\delta}{\alpha-\delta} y_{t-1} = 0 \qquad (4)$$

Equation (4) is the first order linear homogenous difference equation whose genral solution is.

$$y_t = A\left(-\frac{\delta}{\alpha-\delta}\right)^t = A\left(-\frac{\delta}{\delta-\alpha}\right)^t$$

The arbitrary constant $A = y_0$ should be positive as $y_0 > 0$. since, δ and α are positive, the time path of y_t will be explosive and non-osciliatory if $\delta > \alpha$. On the other hand, if $\delta < \alpha$, $\left(\frac{\delta}{\delta-\alpha}\right) < 0$, and the time path of y_t will oscilliate.

2.5. Summary :

In the unit, the concepts and solutions of differential equation and difference equation have been discussed elaborately. The first order differential equation can be solved in two ways. First, with constant coefficient and constant term and second, with variable coefficient and variable term. Moreover, in dynamic stability of market and in Harrod-Domars' simple growth process, the first order differential equation is used. In the analysing of cobweb market model and lagged market model and Harrods' model of growth, the differential equation is not used. Here, as there is time lag, the first order difference equation is used. Moreover, the concept of difference equation and its solution has also been explained in this unit.

2.6: Additional Reading :

1. Chiang, A.C, : "Fundamental Methods of Mathematical Economics", Mc Grow Hill.

2. Baruah, S, "Basic Mathematics and its Economic Application", Mc Millan.

2.7 : Self-Assessment Test :

1. Solve the following differential equations :

a) $\frac{dy}{dx}+8y=4$ given $y(0)=2$. b) $3\frac{dy}{dx}+6y=10$ given $y(0)=4$.

c) $4\frac{dy}{dx}+8xy=4x$ given $y(0)=6$.

2. Give the demands and supply models

$$Q_d = a - dp - \beta\frac{dp}{dt} \qquad (a,b>0)$$
$$Q_s = bp \qquad (b>0)$$

Obtain time path of price p, if rate of change in price over time is directly proportional to excess demand, the adjustment coefficient being $\lambda(\lambda > 0)$. Obtain the restriction on β for dynamic stability.

3. Analyse the following market model for stability

(a) $Q_d = 10-2p$
$Q_s = -8+4p$
$\frac{dp}{dt} = 2(Q_d - Q_s)$

(b) $Q_d = 8-2p$
$Q_s = -8+2p$
$\frac{dp}{dt} = 2(Q_d - Q_s)$

4. Solve the following difference equations :

a) $y_{t+1}-4y_t=12$, with $y_0=8$ b) $y_{t+1}+3y_t=10$, with $y_0=20$

5. Given the demand and supply function for cobweb model.

$$Q_{dt} = 10-2p_t \qquad Q_{st} = -5-3p_{t-1}$$

Find intertemporal equilibrium price and also determine whether you will get stable equilibrium.

6. In a Cobweb Model,

$$Q_{dt} = a - cp_t \quad (a,c>o)$$
$$Q_{st} = -b + dp_{t-1} \quad (b,d>o)$$
$$Q_{dt} = Q_{st}$$

Obtain the time path of and analyse the condition for its convergance.

●●●

UNIT 3

OPTIMIZATION WITH INEQUALITY CONSTRAI

Structure

3.1 INTRODUCTION

The search for the best use or the optimum use has intrigued economists throughout the ages. A rigorous approach to these problems had to wait until economists and mathematicians came together to develop programming method. The linear programming is a technique for selecting the best alternative from a set of feasible alternatives, in situations in which the objective function and the constraints are expressed as linear functions. In a real sense, programming problems are concerned with an efficient use or collection of limited resources to fulfill the desired objectives. The unit provides an idea of general formulation of production, diet, transportation problem, Simplex method and concept of duality also.

3.2 OBJECTIVES

This unit is concerned with the concept of linear programming and graphic method of solution of linear programming problems. On reading this unit, you should be able to:

- know the meaning of linear programming problem.
- solve linear programming problem graphically.
- appreciate the importance of linear programming in economics.
- formulate a linear programming problem.
- solve linear programming problem by simplex method.
- locate the difference between graphic method and simplex method.
- explain the steps involved in simplex method of solution.
- appreciate the importance of simplex method of solution.

3.3 CONCEPT OF LINEAR PROGRAMMING

Linear programming is purely a mathematical technique for the analysis of optimum decisions subject to certain constraints in the form of linear inequalities. It deals with the optimization (maximization or minimization) of a function subject to a set of linear inequalities and/ or inequalities known as constraints. The objective function may be profit, cost, production capacity or any other measure of effectiveness, which is to be obtained in the best possible manner. The constraints may be imposed by different sources such as market demand, production processes and equipment, storage capacity, raw material availability, etc. By the term linearity in mathematics means an expression in which the variables do not have powers. Simply a relationship is said to be linear if it gives straight line when plotted in graph. Linear programming technique is successfully used in the field of military, industry, economics, transportation system, health sector and even behavioral and social sciences.

L.V. Kantorovich, a Russian mathematician first formulated the linear programming technique. Several economists like Koopmans, Dorfman, Solow, Cooper have also contributed to the development of the technique. But it was mathematician George Dantzig who first developed the general computational technique, the simplex method (which is still considered as the most powerful and efficient technique), in 1947 while working on a project for U.S. Air Force is called as the father of linear programming technique.

Requirement of a Linear Programming Problem:

The basic objective of the use of linear programming technique is to make optimum use of limited resources. If resources were not limited perhaps the need of management technique like this would not arise. However, the application of linear programming technique rests on certain requirements:

1. A set of non-negativity constraints:

This condition is required because we are not interested in getting negative solutions, e.g. negative output or negative price do not have any meaning.

2. A set of linear constraints:

This represents the side conditions or the limitations or constraints involved in the solution of the problem. Such constraints are usually due to technological limitations, or resource limitations, or capacity limitations, or time limitations etc.

3. An objective function:

A well-defined objective function is required either to maximize or to minimize subject to the above constraints.

Applications of Linear Programming:

The linear programming technique can be successfully applied to a wide variety of problems. Some of these are discussed below:

1. Military problems: This technique is used in military planning problems.

2. Manufacturing problems: Linear programming technique is used to find out the number of items of each type that should be produced so as to maximize profit.

3. Production problems: It is used to decide the production schedule to satisfy the market demand and to minimize the labour cost, storage cost, etc.

4. Purchasing problems: Linear programming technique is also applied to minimize the cost of production in processing of goods purchased from outside and varying in quantity, quality, and prices.

5. Diet problems: This technique is used in the preparation of hospital diets, even at household levels to fix the minimum requirement of nutrients, subject to the availability of foods and their prices.

6. Transportation problems: Linear programming is also used to determine the optimum amount of goods to be transported from each warehouse to each of the retail stores in order to minimize the total costs of goods transportation. It can also be used in case of passenger transportation to minimize operation cost.

7. Job assigning problems: It is also used to assign jobs to workers for maximum effectiveness and optimum results subject to restrictions of wages and other costs.

Basic Concepts:

Objective Function:

In every linear programming problem, there always exists a definite objective with a set of constraints. The objective of the linear programming problem may be maximization of profit, revenue, sales or output, or minimization of costs such as production, diet or in transportations, etc. It is generally expressed in terms of linear equations with the dependent variables like profit, revenue, output, sales, costs etc., on the left hand side of the equation and other relevant independent variables on the right hand side of the equation. Such functional relationship with a definite objective of maximization or minimization is known as the "objective function" of the linear programming problem. The objective function should be expressed as a linear function of the decision variables. In short, objective function which is also known as the criterion function, describes the "determinants of the quantity to be maximized or to be minimized". If for example, the objective of a firm is to maximize output or profit, then this is the objective function of the firm. An objective function has two parts: the primal and the dual. If the primal of the objective of the firm is to maximize output, then its dual will be the minimization of the cost.

Constraints:

The maximization of the objective function is subjected to certain limitations, which are called constraints or restrictions. The constraints or restrictions are the limitations or bounds imposed on the solution of the problem. For example, if a firm possesses a maximum amount of Rs. 60,000 for investment, then the firm cannot spend/invest more than this amount, which is obviously a limitation on the part of the said firm. Such type of limitation is called as "investment constraint" of the firm. Similarly, the firm may have storage problem, say cannot store more than 60 units of its product. This is another limitation o the firm and we can call this as the "space constraint."

Constraints are also called as inequalities because they are generally expressed in terms of inequalities. In our example, investment constraint is expressed as less than or equal to Rs. 60,000 (i.e., £60,000) and the space constraint is expressed as less than or equal to 60 (i.e., £60).

Feasible Region:

Feasible region is that region where all the constraints are satisfied. All feasible solutions will lie within the feasible region. However there are a large number of feasible solutions within the feasible region. So, problem arises how to choose the optimal solution for a particular linear programming problem. In the graphical methods, the corner points of the feasible regions are considered to get optimum solution.

Iso-cost and Iso-profit Line:

The concept of iso-cost and iso-profit lines have been playing very important role in the solution of linear programming problem through graphical method. The term "iso" means equal and thus the iso-cost and iso-profit lines show equal amount of cost and profit respectively. . Such profit and cost functions are expressed in the form of linear equation of first degree. Therefore, they represent straight lines.

Such iso-cost and iso-profit lines are drawn through the corner points of the feasible region. Thus, we can get a set of parallel iso-profit or iso-cost lines in the graph paper. In case of iso-profit lines, the outer most line from the origin "O" is selected for the optimum solution, as it will show the maximum profit. However, it must also touch one of the corner points of the feasible region. Similarly, in case of iso-cost lines the near most such line is selected for optimum solution as it shows the minimum costs.

Feasible Solution:

The feasible solution can be defined as a point which specifies such values to all the variables involved in the objective function of the problem which would satisfy both types of constraints: structural and non-negativity.

Optimum Solution:

The best of all feasible solutions is the optimum solution for a linear programming problem. In other words, the optimal solution is the best of all feasible solutions. If a feasible solution maximizes or minimizes the objective function, then it is an optimum or optimal solution. For example, if the objective function of a businessman is to maximize profit by selling a combination of two goods, viz., radio and T.V., then the optimal solution will be that combination of radio and T.V. that maximizes the profit of the businessman. On the other hand if the objective of the businessman is to minimize the cost by the choice of a process or combination of processes, then the process or combination of processes, which actually minimizes the cost, will represent the optimum solution. The optimum solution will lie within the feasible solution.

Non-negativity Constraints:

In linear programming problem a set of non-negativity constraints are taken in to consideration. This non-negativity constraints express the necessity that the level production, price, cost of commodities or transportation cannot be negative, since in economics, negative quantities do not carry any sense. These non-negative constraints can be expressed as: $X \geq 0$, $Y \geq 0$ and so on.

Check Your Progress 1:

Mark as True(✓) or False (X):

(a) An LPP can have only two decision variables. ☐

(b) All constraints in an LPP as well as its objective function must be linear in nature. ☐

(c) Every LPP has a unique optimal solution. ☐

(d) For an n variables LPP, there must be an equal number of constraints. ☐

3.4 General Formulation of Production Problems :

In case of production, when a producer produces a set of products, faces a number of technical and physical constraints such as - constraint of availability of inputs, maximum capacity of production units or machines, availability of skilled labour supply etc. The objective of a producing firm is the maximization of profit but its decision of product-mix i.e. amount of output of jointly produced products depends on the rate of profit of the individual product and on the constraints faced by the firm in producing the products.

Let, $x_1, x_2, \ldots, x_m$ be the output of the 'm' products produced jointly by a firm; $r_1, r_2, \ldots, r_n$ be the availability of 'n' inputs used by the firm in the production of 'm' products; $p_1, p_2, \ldots, p_m$ represents the rate of profit per unit of the 'm' products; a_{ij} represents the amount of ith input required to produce one unit of the jth product.

Then the production problem can be formulated as –Maximise the profit

$$\pi = p_1x_1 + p_2x_2 + \ldots\ldots\ldots\ldots + p_mx_m \quad \text{(Objective Function)}$$

$$\text{or } \pi = p_jx_j, \quad j = 1, 2, \ldots, m \quad \text{Subject to}$$

$$a_{11}x_1 + a_{12}x_2 + \ldots\ldots\ldots\ldots + a_{1m}x_m \le r_1$$

$$a_{21}x_1 + a_{22}x_2 + \ldots\ldots\ldots\ldots + a_{2m}x_m \le r_2$$

....

....

$$a_{n1}x_1 + a_{n2}x_2 + \dots\dots\dots\dots + a_{nm}x_m \le r_n$$

Or

$$a_{ij}x_j \le r_i, \qquad i = 1, 2, \dots, n$$
$$x_j \ge 0, \qquad j = 1, 2, \dots, m$$

We can represent the above formulation in matrix form

Maximise profit

$$\pi = p'x \text{ subject to } \quad Ax \le r, \quad x \ge 0$$

where

$$p = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_m \end{bmatrix}; \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}; \quad r = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix} \text{ and } A_{n \times m} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}_{n \times m}$$

3.5 General Formulation of diet Problems :

The diet management at the household level can be made with the help of linear programming technique in order to minimise the cost of diet without compromising the minimum daily requirements of various nutrients to maintain good health. For example, the consumer has to choose a combination of 'm' commodities to minimise the cost of diet satisfying 'n' inequality constraints.

Let, $x_1, x_2, \dots, x_m$ be the 'm' food items; $r_1, r_2, \dots, r_n$ are the minimum daily requirement of nutrients; a_{ij} is the amount of ith nutrient in per kg of jth food and $p_1, p_2, \dots, p_m$ are the prices of 'm' food items.

Then the diet problem can be formulated as --

Minimise

$$C = p_1x_1 + p_2x_2 + \dots\dots\dots\dots + p_mx_m$$

Subject to

$$a_{11}x_1 + a_{12}x_2 + \dots\dots\dots + a_{1m}x_m \geq r_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots\dots\dots + a_{2m}x_m \geq r_2$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

$$a_{n1}x_1 + a_{n2}x_2 + \dots\dots\dots + a_{nm}x_m \geq r_n$$

Or

$$a_{ij}x_j \geq r_i, \qquad i = 1, 2, \dots, n$$
$$x_j \geq 0, \qquad j = 1, 2, \dots, m$$

We can represent the above formulation in matrix form

Minimise

$$C = p'x \quad \text{subject to} \quad Ax \geq r, \quad x \geq 0$$

where

$$p = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_m \end{bmatrix}; \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}; \quad r = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix} \text{ and } A_{n \times m} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}_{n \times m}$$

3.6 General Formulation of transportation Problems :

The linear programming can be used in the problem of goods transport and even in passenger transport. In problem of transportation of goods, the objective is to determine the optimum amount of goods to be transported from each warehouse to each of the retail stores in order to minimise the total costs of transportation. Suppose, goods are transported from 'm' warehouses to 'n' retail shops. $w_1, w_2, \dots, w_m$ are the maximum capacities of 'm' warehouses, $r_1, r_2, \dots, r_n$ are the minimum daily requirement of 'n' retail shops; C_{ij} is the cost of transportation per unit of goods from ith warehouse to the jth retail shop and x_{ij} represent the amount of the good transported from ith warehouse to jth retail shop.

The transportation problem can be formulated as --

Minimise

$$C = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} \qquad \text{(Objective Function)}$$

Subject to $C = \sum_{i=1}^{m} x_{ij} = r_j \qquad j = 1, 2, \ldots, n$

i.e. $x_{11} + x_{21} + \ldots\ldots\ldots\ldots + x_{m1} = r_1$

$x_{12} + x_{22} + \ldots\ldots\ldots\ldots + x_{m2} = r_2$

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$x_{1n} + x_{2n} + \ldots\ldots\ldots\ldots + x_{mn} = r_n$

and

$$\sum_{j=1}^{n} x_{ij} = w_i \qquad i = 1, 2, \ldots, m$$

i.e. $x_{11} + x_{12} + \ldots\ldots\ldots\ldots + x_{1n} = w_1$

$x_{21} + x_{22} + \ldots\ldots\ldots\ldots + x_{2n} = w_2$

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$x_{m1} + x_{m2} + \ldots\ldots\ldots\ldots + x_{mn} = w_m$

$x_{ij} \geq 0, \qquad i = 1, 2, \ldots, m; \qquad j = 1, 2, \ldots, n$

3.7 GRAPHIC SOLUTION

Two methods are generally available for solving the linear programming problems. One is the simple graphical method and the other is the mathematical method, known as the simplex method. The graphical method is simple and is presented on a two dimensional diagram. It is suitable when one considers only two variables and when we have to consider more than two variables; the graphical solution becomes extremely difficult to draw any conclusion. In such cases, simplex method is extremely useful. The graphical solution involves two steps: (1) the graphical determination of the region of feasible solution and (2) the graphical presentation of the objective function.

Let us consider the following problem to have an idea about the graphical method of solving the linear programming problems:

A manufacturer produces nuts and bolts for some industrial machinery. It takes 1 hour of work on machine A and 3 hours on machine B to produce a package of nuts while it takes 3 hours on machine A and 1 hour on machine B to produce a package of bolts. He earns profit of Rs. 2.50 per package of nuts and Rs. 1.00 per package of bolts. How many package of each should be produced so as to maximize his profit if he operates his machine almost 12 hours a day. What is the value of maximum amount of profit?

If this is the nature of the problem, we have to translate first the problem in to mathematical form by finding the objective function and the other necessary constraints.

To solve the problem let us consider that the manufacturer produces X packages of nuts and Y packages of bolts to maximize his profit (p). Therefore our objective function becomes:

p= 2.50X +1.00Y, which is to be maximized subject to the following constraints:

$X+3Y \leq 12$

$3X+Y \leq 12$

Non-negativity Constraints:

X ³ 0, Y ³0

Now to represent the above inequalities in graph paper, we have to transform them in the following form:

$X+3Y=12$

$3X+Y=12$

The next step is to find out the coordinates for the both equation by assuming the value of X and Y as 0.

Therefore,

when X=0, Y=4; so A (0,4).

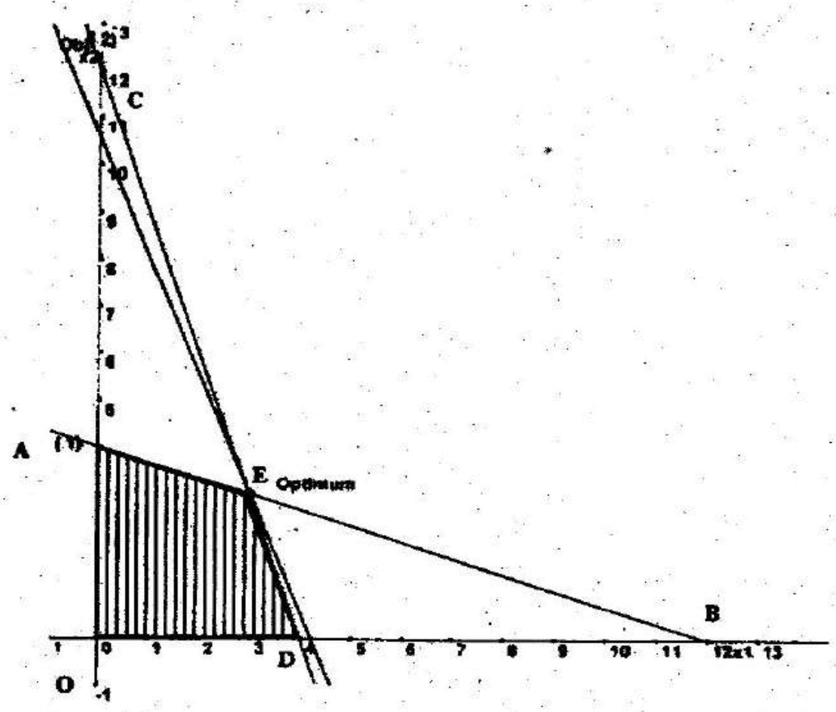
when Y=0, X=12; so B (12,0).

Similarly, when we take the 2nd equation: 3X+Y=12, we get the following coordinates:

when X=0, Y=12; C (0,12)

when Y=0, X=4; D (4,0).

Now we are in a position to plot the above points on the graph and by plotting it in the graph we get OADE is the feasible region, which will yield optimum solution. There are four corner points O (0,0), A (0,4), E (3,3) and D (4,0).



The next step is to draw the iso-profit line through these corner points to choose the outer most iso-profit line, which yields maximum profit. We can ignore the point O (0,0), which implies:

$$p= 2.50X+ 1.00 Y$$

Iso-profit line passes through A (0,4):

Therefore, by substituting X=0 and Y=4 in p= 2.50X+ 1.00 Y, we get:

p= 2.50 X 0+ 1.00 X 4

= 0+ 4.00

= 4.00

∴ 2.50X+ 1.00 Y= 4.00

Now, when Y=0, substituting it in 2.50X+ 1.00 Y=4.00, we get X=1.6. Thus, when X= 0, Y=4; A (0,4) and when Y=0, X=1.6; F (1.6,0).

Iso-profit line passes through point E (3,3):

p= 2.50 X 3+ 1.00 X 3 = 10.50

∴ 2.50X+ 1.00 Y = 10.50

Thus when, X = 0, Y = 10.50; H (0,10.5) and when Y=0, X = 4.2; I (4.2,0).

Iso-profit line passes through point D (4,0):

p = 2.50 X + 1.00 Y

∴ 2.50 x 4 + 1.00 x 0 = 10

Thus when, X = 0, Y = 10; J (0,10) and when Y=0, X = 4; D (4,0).

After plotting the iso-profit lines in the graph through the corner points of the feasible region, it is found that the iso-profit line: 2.50X + 1.00Y = 10.50 is the outer most iso-profit line from the origin "O" indicating maximum profit of Rs. 10.50. This particular iso-profit line passes through the corner point E (3,3). It indicates that the manufacturer will produce 3 packages of nuts and 3 packages of bolts to maximize his profit at Rs. 10.50.

3.8 SIMPLEX METHOD

The simplex algorithm is an iterative method for finding the optimal solution of a linear programming problem. If a feasible solution to the linear programming problem exists, it is located at a corner point of the feasible region determined by the constraints. The simplex method, following the iterative search, locates this optimal solution from among the set of feasible solutions to the given problem. The simplex method considers only those feasible solutions (not all), which are located in the corner points. By this method we can consider a minimum number of feasible solutions to obtain the optimal solution. This method also tells us whether a given solution is optimal or not. The simplex method proceeds by preparing a series of tables called simplex tableaus.

Now, we shall acquaint you with the simplex procedure of solution with the help of an example.

Example:

Maximize $\pi = 40x + 35y$

Subject to $2x + 3y \leq 60$

$4x + 3y \leq 96$

$x, y \geq 0$

Solution:

The first step in applying simplex method is to standardize the given problem. For this, inequalities in the constraints are converted into equality constraints. This can be done with the help of slack variables. Let us take the first constraint $2x + 3y \leq 60$. This inequality implies that the sum of the terms of the inequality sign is less than or equal to 60. Stated in another way this equation also mean that there exist a number say $S_1 \geq 0$, which when added to the terms of the left hand side of the inequality will convert it into equality. This additional number is known as slack variable.

Applying slack variables, we convert the constraints as below:

$$2x + 3y + S_1 = 60$$

$$4x + 3y + S_2 = 96$$

and the objective function as

Maximize $\pi = 40x + 35y + 0.S_1 + 0.S_2$

The converted constraints can be written in the vector notation as:

$$\begin{bmatrix}2\\4\end{bmatrix} x + \begin{bmatrix}3\\3\end{bmatrix} y + \begin{bmatrix}1\\0\end{bmatrix} S_1 + \begin{bmatrix}0\\1\end{bmatrix} S_2 = \begin{bmatrix}60\\96\end{bmatrix}$$

Let, $\begin{bmatrix}2\\4\end{bmatrix} = P_1$ (Structural vector)

$\begin{bmatrix}3\\3\end{bmatrix} = P_2$ (Structural vector)

$\begin{bmatrix}1\\0\end{bmatrix} = P_3$ (Identity vector)

$\begin{bmatrix}0\\1\end{bmatrix} = P_4$ (Identity vector)

$\begin{bmatrix}60\\96\end{bmatrix} = P_0$ (Identity vector)

Thus the whole problem reduces to:

Maximize $\pi = 40x + 35y + 0.S_1 + 0.S_2$

Subject to $P_1 x + P_2 y + P_3 S_1 + P_4 S_2 = P_0$

and $x, y, S_1, S_2 \geq 0$

This is the standardized form of the given problem.

The simplex tableau is formed in a particular way as explained below:

Stages	$C_j \rightarrow$			0	0	40	35	Replacement Ratio
	$\downarrow$		P_0	P_3	P_4	P_1	P_2	
I	0	P_3	60	1	0	2	3	$\frac{a_{30}}{a_{31}} = \frac{60}{2} = 30$
	0	P_4	96	0	1	4	3	$\frac{a_{40}}{a_{41}} = \frac{96}{4} = 24$
	Z_j		0	0	0	0	0	
	$Z_j - C_j$			0	0	$\boxed{-40}$	−35	

P_0 vector appears first, followed by the basis (identity) vectors, viz. P_3, P_4, followed by the structural vectors P_1 and P_2. In the first row of the table (C_j) we write the coefficients of the vector of the objective function in the standardized form, which is required to be maximized, following the order described in the table. In the first column of the table (C_j), we write the coefficients of the basis vectors at the first stage.

Next stage is to calculate elements of Z_j raw. These are calculated as the summation of products of elements of each column vector with corresponding elements of C_j column. For example, elements of P_3 column are (1, 0), while corresponding elements of C_j column are (0, 0). Hence Z_j for P_3 column is $(0 \times 1) + (0 \times 0) = 0$

Next step is to formulate $Z_j - C_j$ row by subtracting each value of C_j given in the first row of the table from the corresponding values of Z_j.

Before going to the next table, following test is used to determine whether the solution of the given linear programming problem is an optimal feasible solution, or whether it is necessary to make further iterations. The test is performed in the following way:

(a) If all $Z_j - C_j \geq 0$, an optimal solution has been obtained. Hence no further iterations are necessary.

(b) If $Z_j - C_j < 0$, for some columns then

(i) If all elements of those columns for which $Z_j - C_j < 0$ possess negative values, the solutions will be infinite.

(ii) If some of the elements of those columns (for which $Z_j - C_j < 0$) possess positive values, further iterations are necessary to achieve the optimal solution.

By applying the above test in our example it is found that in the first stage $Z_j - C_j$ values are less than 0 for P_1 and P_2. Hence further iteration is needed to arrive at the optimal solution.

Now, structural vectors are used to replace the basis vectors in turn. We will have to decide which vector is replacing and which vector is to be replaced,

Rule for locating the replacing vector

Replacing vector will be that structural vector which has highest negative $Z_j - C_j$ value amongst them. In the present example P_1 is the replacing vector.

Rule for locating the vector to be replaced

The replaced vector is determined by means of finding the ratio of each element in P_0 vector to the corresponding elements of the replacing vector, P_1 (in this example). The basis vector associated with the smallest positive ratio would be the vector to be replaced. In this example, P_4 will be the vector to be replaced. For this replacement ratios are to be calculated.

Now, we can move to stage II

In the second table, the replacing vector P_1 will be introduced in place of the vector to be replaced P_4 in the second column of the second table.

We prepare the second table

Stages	$C_j \rightarrow$			0	0	40	35	Replacement Ratio
	$\downarrow$		P_0	P_3	P_4	P_1	P_2	
II	0	P_3	12	1	-0.5	0	1.5	$\frac{a_{30}}{a_{32}} = \frac{12}{1.5} = 8$
	40	P_1	24	0	0.25	1	0.75	$\frac{a_{10}}{a_{12}} = \frac{24}{0.75} = 32$
	Z_j		960	0	10	40	30	
	$Z_j - C_j$			0	10	0	-5	

The elements of this table are calculated in the following ways:

(a) The elements in the raw of the new vector P_1 in the second table are obtained by dividing each element of P_4 vector in the first table by the corresponding element of vector P_1 in the first table. Therefore, elements of P_1 row in the second table are (calculated from first table):

$$\frac{a_{40}}{a_{41}} = \frac{96}{4} = 24$$

$$\frac{a_{43}}{a_{41}} = \frac{0}{4} = 0$$

$$\frac{a_{44}}{a_{41}} = \frac{1}{4} = 0.25$$

$$\frac{a_{41}}{a_{41}} = \frac{4}{4} = 1$$

$$\frac{a_{42}}{a_{41}} = \frac{3}{4} = 0.75$$

(b) The elements of the remaining rows are determined by the following rule:

Elements of P_3 row in table II are (calculated from table I)

$$a_{30} - \left(\frac{a_{31}}{a_{41}} \times a_{40}\right) = 60 - \left(\frac{2}{4} \times 96\right) = 12$$

$$a_{33} - \left(\frac{a_{31}}{a_{41}} \times a_{43}\right) = 1 - \left(\frac{2}{4} \times 0\right) = 1$$

$$a_{34} - \left(\frac{a_{31}}{a_{41}} \times a_{44}\right) = 0 - \left(\frac{2}{4} \times 1\right) = -0.5$$

$$a_{31} - \left(\frac{a_{31}}{a_{41}} \times a_{41}\right) = 2 - \left(\frac{2}{4} \times 4\right) = 0$$

$$a_{32} - \left(\frac{a_{31}}{a_{41}} \times a_{42}\right) = 3 - \left(\frac{2}{4} \times 3\right) = 1.5$$

Once you fill up all the cells of the table repeat the procedure outlined above, i.e., next step is to formulate Z_j and $Z_j - C_j$ rows.

$Z_j - C_j$ value is less than 0 for P_2. Hence further iteration is needed to arrive at the optimal solution.

We will have to decide which vector is replacing and which vector is to be replaced.

Now, we can move to stage III

In the third table, the replacing vector P_2 will be introduced in place of the vector to be replaced P_3 in the second column of the second table.

We prepare the third table

Stages	$C_j \rightarrow$			0	0	40	35	Replacement Ratio
	$\downarrow$		P_0	P_3	P_4	P_1	P_2	
III	35	P_2	8	0.67	−0.33	0	1	
	40	P_1	18	−0.50	0.50	1	0	
	Z_j		1000	3.45	8.45	40	35	
	$Z_j - C_j$			3.45	8.45	0	0	

The elements of this table are calculated in the following ways:

(a) The elements in the raw of the new vector P_2 (newly introduced) in the third table are obtained by dividing each element of P_3 vector in the second table by the corresponding element of vector P_2 in the second table. Therefore, elements of P_2 row in the third table are (calculated from second table):

$$\frac{a_{30}}{a_{32}} = \frac{12}{1.5} = 8$$

$$\frac{a_{33}}{a_{32}} = \frac{0}{1.5} = 0$$

$$\frac{a_{34}}{a_{32}} = \frac{-0.5}{1.5} = -0.33$$

$$\frac{a_{31}}{a_{32}} = \frac{0}{1.5} = 0$$

$$\frac{a_{32}}{a_{32}} = \frac{1.5}{1.5} = 1$$

(b) The elements of the remaining rows are determined by the following rule:

Elements of P_1 row in table III are (calculated from table II)

$$a_{10} - \left(\frac{a_{12}}{a_{32}} \times a_{30}\right) = 24 - \left(\frac{0.75}{1.5} \times 12\right) = 18$$

$$a_{13} - \left(\frac{a_{12}}{a_{32}} \times a_{33}\right) = 0 - \left(\frac{0.75}{1.5} \times 1\right) = -0.50$$

$$a_{14} - \left(\frac{a_{12}}{a_{32}} \times a_{34}\right) = 0.25 - \left(\frac{0.75}{1.5} \times (-0.5)\right) = 0.50$$

$$a_{11} - \left(\frac{a_{12}}{a_{32}} \times a_{31}\right) = 1 - \left(\frac{0.75}{1.5} \times 0\right) = 1$$

$$a_{12} - \left(\frac{a_{12}}{a_{32}} \times a_{32}\right) = 0.75 - \left(\frac{0.75}{1.5} \times (1.5)\right) = 0$$

Once you fill up all the cells of the table repeat the procedure outlined above, i.e., next step is to formulate Z_j and $Z_j - C_j$ rows.

None of the $Z_j - C_j$ value is less than 0. Hence further iteration is not needed and an optimal solution has been arrived.

The optimal solution is read from the last table. The value of x is read from the value associated with P_1 in P_0 column. In the same way value of y is read from the value associated with P_2 in P_0 column.

$x = 18, \; y = 8.$

This solution yields the objective function value

$$\pi = (40 \times 18) + (35 \times 8) = 1000$$

Check Your Progress 1

List the steps involved in simplex method to a maximization problem.

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3.9 Concept of Duality :

Every linear programming problem has a corresponding dual problem. A maximisation problem has a corresponding minimization problem and vice-versa. The original problem is called the primal and the corresponding problem is called the dual. The optimal solution of primal serves as the optimal solution of the dual.

Rules of transformation from primal to dual problem :

(1) The direction of optimization is reversed. The minimisation program becomes maximisation program and vice-versa.

(2) The inequality signs of the constraints are reversed i.e. greater than equal sign $(\geq)$ becomes less than equal $(\leq)$ in dual constraints and vice-versa.

(3) The coefficients of choice variables in the objective function of the primal program becomes the constants of the constraints of dual program.

(4) The constants of the constraints of the primal program becomes the coefficients of the dual choice variables in the objective function.

(5) The rows of the coefficient matrix of the constraints in the primal program are transposed to columns for the coefficient matrix of constraints in the dual program.

(6) The primal choice variable x_i is replaced by dual choice variable y_j

If the primal program is -

Primal

Maximize $\pi = r_1 x_1 + r_2 x_2$

Subject to

$$a_{11}x_1 + a_{12}x_2 \le b_1$$

$$a_{21}x_1 + a_{22}x_2 \le b_2$$

$$a_{31}x_1 + a_{32}x_2 \le b_3$$

in matrix form

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \le \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$x_1,\ x_2 \ge 0$$

then the dual program will be

Dual

Minimize $\pi^* = b_1 y_1 + b_2 y_2 + b_3 y_3$

Subject to

$$a_{11}y_1 + a_{21}y_2 + a_{31}y_3 \ge r_1$$

$$a_{12}y_1 + a_{22}y_2 + a_{32}y_3 \ge r_2$$

in matrix form

$$\begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \ge \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$$

$$y_1,\ y_2,\ y_3 \ge 0$$

3.10 LET US SUM UP

In this unit you have learnt the meaning and graphic procedure of solution of linera programming problems. Linear programming is purely a mathematical technique for the analysis of optimum decisions subject to certain constraints in the form of linear inequalities. It deals with the optimization (maximization or minimization) of a function subject to a set of linear inequalities and/ or inequalities known as constraints. Two methods are generally available for solving the linear programming problems. One is the simple graphical method and the other is the mathematical method, known as the simplex method. The graphical method is simple and is presented on a two dimensional diagram. It is suitable when one considers only two variables. But when we have to consider more than two variables; the graphical solution becomes extremely difficult to draw any conclusion. In such cases, simplex method is extremely useful. The graphical solution involves two steps: (1) the graphical determination of the region of feasible solution and (2) the graphical presentation of the objective function.

Further, we have illustrated the solution of a linear programming problem by simplex method. Simplex method is an iterative procedure for finding, in a systematic manner, the optimal solution to a linear programming problem.

You have learnt to maximize an objective function subject to inequality constraint by simplex method.

3.11 KEY WORDS

Linear Programming: Linear programming is purely a mathematical technique for the analysis of optimum decisions subject to certain constraints in the form of linear inequalities.

Simplex Method: An iterative procedure for finding, in a systematic manner, the optimal solution to a linear programming problem.

Simplex Algorithm: An iterative procedure for finding, in a systematic manner, the optimal solution to a linear programming problem.

Slack Variable: It takes up any slack between the left and the right hand side of the inequality upon being converted into equation.

3.12 TERMINAL QUESTIONS

1. Solve the following problem graphically

Maximize $p = x + 4y$

Subject to $2x + 3y \leq 4$

$3x + y \leq 3$

x, y ³ 0

2. Maximize $\pi = x + 4y$

Subject to $2x + 3y \leq 4$

$3x + y \leq 3$

$x, y \geq 0$

3. Maximize $\pi = 4x + y$

Subject to $4x + 6y \leq 8$

$6x + 2y \leq 6$

$x, y \geq 0$

3.13 ADDITIONAL READINGS

Baruah Srinath (2001): Basic mathematics and its Application in Economics, Macmillan, Calcutta.

Chiang Alpha C (1984): Fundamental Methods of Mathematical Economics, McGraw-Hill, Singapore.

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UNIT 4

GAME THEORY

Structure :

4.1 INTRODUCTION :

Game theory is a discipline concerned with the decision making in situations where two or more rational players are involved under competitive conditions and with conflicting interests. The prime objective in game theory is to determine the rules of rational behaviour in which the outcomes are dependent on the actions of the interdependent players. Two person zero-sum games, solution of this game with saddle point, without saddle point, dominance rules, nush equilibrium, prisoner's dilema, repeated games are also discussed.

4.2 OBJECTIVES :

This unit is concerned with the concept game theory and the variants of game. We shall also acquaint you with the importance and limitations of game ant its application in duopoly market. Further, this unit is concerned with the concept two-person zero-sum game and its solution. On reading this unit, you should be able to:

- understand the concept of game theory.
- know the basic terminologies involved in game theory.
- locate the differences of different types of games.
- refine your analytical tools to present the pay-offs of the players in matrix form.
- appreciate the importance of game theory in economics.
- understand the concept of two-person zero-sum game.
- solve two-person zero-sum game.
- locate the differences of games with pure strategies and mixed strategies.
- refine your analytical tools to solve game graphically.
- appreciate the importance of game theory in economics.

4.3 CONCEPT OF GAME :

The theory of game was developed by Von Neumann in 1928. It was, however, only after 1944 Von Neumann and Oskar Morgenstern published their now well known "Theory of Games and Economic Behaviour" that the theory received the proper attention. A more recent and in-depth presentation of the game theory with economic applications to economic problems is found in the works of M.J. Osborne and A. Rubinstein (1994). The other important contributors to the game theory were John Nash and Thomas Schelling. In general, game theory is concerned with the choice of an optimal strategy in conflicting situations. It deals with the mathematical analysis of competitive problems and is based on the minimax strategy put forwarded by Von Neumann, which implies that each competitor will act so as to minimize his maximum loss (or maximize his minimum gain). In economics game theory can help a doupolist or an oligopolist to choose the course of action that maximizes the benefit or profit after considering all the possible actions of its rival. In both form of markets it is very difficult to arrive at a determinate solution as the interests and strategies of the participants are conflicting. Thus game theory, for example, can help an oligopolistic firm to decide: (1) the conditions under which lowering of its price will not result in price war, (2) whether the firm should build excess capacity to discourage the entry of new firm into the industry, even though it may incur lose in the short run; (3) why cheating leads to break up of cartels. Game theory can be of great use in the analysis of conflicting situations like these. *In short, game theory can be defined as the modeling of economic decisions by games to gain competitive advantage over the rival or to minimize the potential harm from a strategic move by the rival, whose outcomes depends on the decisions taken by the two or more agents or players, each having to make decisions without knowing what strategies each of them are following.*

4.4 ASSUMPTIONS OF GAME THEORY :

The theory of game is based on certain assumptions, which are mentioned below:

(a) There is finite number of participants called players.

(b) A list of finite or infinite number of possible courses of action is available to each player. The list need not be same for each player. Such a game is said to be in normal form.

(c) A play is played when each player chooses one of his courses of action, the choices are assumed to be made simultaneously, so that no player knows his opponents choice until he has decided his own course of action.

(d) Every play is associated with an outcome, known as payoff, which determines a set of gains, one to each player. Here a loss is considered as a negative gain. Thus, after each play of the game, one player pays to others an amount determined by the courses of actions chosen.

(e) All players act rationally and intelligently. It means that each player has a consistent ranking over the all the possible outcomes and calculates the strategy that serves his interest best. Thus they are perfect calculators and flawless followers of best strategies.

(f) Each player attempts to maximize his gain or minimize loss.

(g) Each player makes individual decision without direct communication.

(h) Each player knows complete relevant information.

4.5 BASIC CONCEPTS :

Before discussing about game theory in detail, let us first throw light on the some basic concepts of game theory:

Every game theory involves players, strategies, and payoffs. Let us first define these three:

Player:

Each of the participants in a game is called a player. For example in a duopoly there are two players who can participate in the game.

Play:

In game theory, a play results when each player has chosen a course of action.

Strategy:

The decision rule by which a player determines his course of action is called a strategy. Simply strategies are the choices available to the players. It clearly defines specific course of action in value terms for the policy variable. For example, a strategy may consist of setting a price of Rs. 5.00, spending Rs. 3000 on advertising, making a change in the packaging of the product, and selling the product in the super markets. Another strategy may involve keeping price unchanged, spending Rs. 1000 on advertisement and spending Rs. 3000.00 on research and development activities for a new product and so on. On the other hand the rivals will take their own course of action as against each of these strategies separately. They may take same course of action or may not take same course of action. However, to reach the decision regarding which strategy is to use, neither player needs to know the other's strategy.

Strategies can be of two types:

(a) Pure Strategy and (b) Mixed Strategy. A strategy is called pure strategy if a player decides to use only one particular course of action during every play. A pure strategy is usually represented by a number with which course of action is associated. No randomness is associated with this strategy. On the other hand, if a player decided in advanced to use all the courses of action or some of available courses of action in some fixed proportion, then the player is said to use mixed strategy. A mixed strategy is a selection among pure strategies with some fixed proportion.

Payoff:

The payoff is the outcome or consequence of each strategy. While taking any strategy by a firm some alternative strategies are available to the competitive firms and payoff is the result of the each of the combination of strategies by the firms. The payoff is usually expressed in terms of the profits or loses. In other words, a quantitative measure of satisfaction a person gets at the end of each play is called a payoff. It is a real-valued function of variables in the game. Let v_i be the payoff to the player p_i, $1 \le i \le n$, is an n-person game.

If $\sum_{i=1}^{n} v_i = 0$, then the game is said to be a non-zero sum game. Payoff may be such that the gains of some players may and may not be direct losses of others players.

On the other hand the payoff matrix may also be considered as a table showing the amounts received by the player named at the left hand side after all possible plays of the game. The player named at the top of the table makes the payment.

For example, if player A has m-courses of action and the player B has n-courses of actions, then a payoff matrix can be constructed as given below:

(a) Row designations for each matrix are the courses of action available to player A.

(b) Column designations for matrix are the courses of action available to B.

(c) With a two-person zero-sum game, the cell entries in B's payoff matrix will be the negative of corresponding entries in A's payoff matrix and the matrices will appear as follows:

Table-1

Payoff Matrix of Player A

		Player B					
		1	2	3	...	j...	n
Player A	1	a_{11}	a_{12}	a_{13}	...	a_{1j}...	a_{1n}
	2	a_{21}	a_{22}	a_{23}	...	a_{2j}...	a_{2n}
	3	a_{31}	a_{32}	a_{33}	...	a_{3j}...	a_{3n}
	⋮	⋮	⋮	⋮	⋮	⋮	⋮
	i	a_{i1}	a_{i2}	a_{i3}	...	a_{ij}...	a_{in}
	⋮	⋮	⋮	⋮	⋮	⋮	⋮
	m	a_{m1}	a_{m2}	a_{m3}	...	a_{mj}...	a_{mn}

Table-2

Payoff Matrix of Player B

		Player B					
		1	2	3	...	j...	n
Player A	1	$-a_{11}$	$-a_{12}$	$-a_{13}$	...	$-a_{1j}$...	$-a_{1n}$
	2	$-a_{21}$	$-a_{22}$	$-a_{23}$	...	$-a_{2j}$...	$-a_{2n}$
	3	$-a_{31}$	$-a_{32}$	$-a_{33}$	...	$-a_{3j}$...	$-a_{3n}$
	⋮	⋮	⋮	⋮	⋮	⋮	⋮
	i	$-a_{i1}$	$-a_{i2}$	$-a_{i3}$	...	$-a_{ij}$...	$-a_{in}$
	⋮	⋮	⋮	⋮	⋮	⋮	⋮
	m	$-a_{m1}$	$-a_{m2}$	$-a_{m3}$	...	$-a_{mj}$...	$-a_{mn}$

Thus the sum of the payoff matrices of the two players A and B are zero. Here, the objective of the players is to determine the optimum strategies that result optimum payoff to each, irrespective of the strategy used by the other. In the above payoff matrices constructed for A and B reflects that whatever is the strategy followed by the Player A, the player B follows its own counter strategy out of the several alternatives available to it so as to minimize his lose. Here negative payoff of the player B suggest that it is just the negative of the payoff matrix A. Usually, we do not construct the payoff matrix for B, as it is just negative of A.

Some Other Important Concepts:

Competitive Game: A competitive situation is called a competitive game if it possesses the following six properties:

1. There are finite numbers of participants. The number of participants is $n \geq 2$. If n=2, it is called a two person game; and if n = 2, then it is called n person game.

2. Each person has a finite number of possible courses of action.

3. Each participant must know the possible courses of action available to others but must not know which of these will be chosen.

4. A play of the game occurs when each player chooses one of his courses of action. The choices are assumed to be made simultaneously, so that no participant no the choice of other until he has decided his own.

5. After all participants have chosen a course of action, their respective gains are finite.

6. The gain of the participant depends upon his actions as well as those of others.

Two-Person Zero Sum Game: A game with two players, where a gain of one player equals the loss to the other player is known as the *two person zero sum game*. In such type of game, interests of the two players are opposed so that the sum of their gains is zero. For example, if two players of chess agree that at the end of the game the loser would pay Rs 100 to the winner of the game, then it would mean a zero sum game since the gain of one is equal to the loss of the other. On the other hand, if there are n players and sum of the game is zero then it is called *n person zero sum game.*

Two person zero sum game is also called rectangular games because their payoff matrix is in the rectangular form. A two person zero sum game exhibits certain characteristics as follows:

1. Only two players participate.
2. Each player has finite number of possible courses of action.
3. Each specific strategy results in a payoff.
4. Total payoff to the two players at the end of each play is zero.

Non Zero Sum Game :

A game is called a non zero sum game if the gains or loses of one firm do not come at the expenses of or provide equal benefit to the other firm. If for example increased advertisement results in the higher profits to both the firms, it is a positive sum game. On the other hand if increased advertisement results in rise in cost than revenue suggesting a declining profit. This is a case of negative sum of game. Both these are the example of non-zero sum game.

Co-operative Games:

Games in which joint-action agreements are enforceable are called cooperative games.

Non-cooperative Games:

Games in which enforcement of joint-action games are not possible and individual must be allowed to act in their own interest are called non-cooperative games.

Uncertainty Model:

The assumption that each firm knows with certainty the exact value of the payoff of each strategy is unrealistic. The most probable situation in real world is that when a firm adopts a particular strategy, may expect a range of results for each counter-strategy followed by its rival, each with an associated probabilities. Therefore the payoff matrix is constructed so as to include the expected value of each payoff. The expected value is the sum of the products of the possible outcomes of a pair of strategies (adopted by the two firms) each multiplied by its probability. Mathematically,

$$E(G_{ij}) = g_{1i}P_1 + g_{2i}P_2 + \ldots + g_{ni}P_n = \sum_{s=1}^{n} g_{si}P_s$$

where g_{si} = the sth of the n possible outcomes of strategy i of firm I (given that Firm II has chosen strategy j)

P_s = the probability of the sth outcome of the strategy i.

Check Your Progress 1

Mark the following statements as True or False:

(a) In a two-person game, both the players should have an equal number of strategies. ☐

(b) Games in which joint-action agreements are enforceable are called cooperative games. ☐

(c) Saddle point is not the point of equilibrium. ☐

4.6 CHARACTERISTICS OF GAME THEORY :

The game theory possess certain characteristics which are mentioned below:

(a) **Chance of strategy:** A game may be game of strategy or game of chance. If in a game activities are determined by skill, it is said to be a game of strategy; and if they are determined by chance, it is a game of chance.

(b) **Number of persons:** A game is called an n-person game if the number of persons playing the game is n. the person means an individual or a group aiming at a particular objective.

(c) **Number of activities:** The number of activities in a game may be finite or infinite.

(d) **Number of alternatives:** Number of alternatives available to each person in a particular activity may also be finite or infinite. A finite game has a finite number of activities, each involving a finite number of alternatives, otherwise the game is said to be infinite.

(e) **Information to the players:** Information to the players about the past activities of other players is completely available, partly available, or not available at all.

4.7 APPLICATION OF GAME THEORY IN DUOPOLY :

Game theory shows the importance to duopolists in finding some way to agree. It helps to explain why duopoly prices tend to be administered in a rigid way. If prices were to change often, tacit agreements would not be found and would be difficult to enforce. A serious analytical difficulty under duopoly arises directly out of a firm's need to take account of its competitor's reaction patterns. When a firm's manager thinks about making a decision, he takes into account the likely response of his competitors to it, but he has to recognize that his competitor, too, is likely to take this interdependence phenomenon into account. The firms, thus, attempt to outguess each other. This leads to interplay of anticipated strategies and counter strategies which leads to a complicated situation. The firm is only led to advance along an infinite sequence of compounded hypotheses about its rival.

There are three ways to solve this confusion and economists have tries to analyse duopoly based on the following:

(a) One approach which businessmen also adopt in practice is to assume that the firms ignore the interdependence for minor day-to-day decisions and recognize it only with regard to a few major policy actions which are very rarely taken up.

(b) Another approach to simplify duopoly analysis is for a firm to anticipate the nature of competitive reactions on the basis of guesswork on past experiences. In such cases, it is possible to take the reaction pattern into account and decide on a strategy, which is optimal in terms of the experiences.

(c) The third approach to solve duopoly behaviour is through game theory. In this approach, the firm is not taken to guess at his opponent's reaction pattern, but instead is thought to calculate the optimal moves of the competitor - his rival's best possible strategies - and prepare his own defences and counter measures accordingly. Some important models of duopoly viz., Cournot model, Stackelberg model etc. can be analyzed in the game format.

4.8 LIMITATIONS OF GAME THEORY

Game theory, which was initially received in literature with great enthusiasm as holding promise, has been found to have a lot of limitations. Some important limitations of the game theory are mentioned bellow:

(a) The assumption that the players have the knowledge about their own payoffs and payoffs of their opponents is rather unrealistic. He can only make a guess of his own and his rival strategies.

(b) As the number of players increases in the game, the analysis of the gaming strategies become increasingly difficult and complex. In practice, there are many firms in an oligopoly situation and game theory can not be very helpful in such situations.

(c) The assumptions of maximin and minimax show that the players are risk-averter and have complete knowledge of the strategies. These do not seem practical.

(d) Rather than each player in an oligopoly situation making under uncertain conditions, the players will allow each other to share the secrets of business in order to work out a collusion and under such situations mixed strategies are not very helpful.

Check Your Progress 2

Mention three limitations of the game theory.

4.9 IMPORTANCE OF GAME THEORY

Game theory as applied to value theory possesses the following merits:

(a) Game theory shows the importance to duopolists in finding some way to agree. It helps to explain why duopoly prices tend to be administered in a rigid way. If prices were to change often, tacit agreements would not be found and would be difficult to enforce.

(b) Game theory also highlights the importance of self-interest in the business world. In game theory, self-interest is routed through the mechanism of economic competition to bring the system to the saddle point. This shows the existence of perfectly competitive market.

(c) Game theory tries to explain how duopoly problem can not be determined.

(d) Game theory has been used to explain the market equilibrium when more than two firms are involved. The solution lies in either collusion or non-collusion. These are known as cooperative non-constant sum game and non-cooperative game respectively.

(e) 'Prisoner's Dilemma" in game theory points towards collective decision-making and the need for cooperation and common rules of road.

(f) The importance of the pay-off values lies in predicting the outcomes of a series of alternative choices on the part of the player. Thus a perfect knowledge of the pay-off matrix to a player implies perfect predictions of all factors affecting the outcome of alternative strategies. Moreover, the minimax principle shows to the player the next course of action, which would minimize the losses if the worst possible situation arose.

(g) Game theory is also helpful in solving the problems of business, labour and management. As a matter of fact, a business always tries to guess the strategy of his opponents so as to implement his plans more effectively.

(h) Last but not the least, there are certain economic problems, which involve risk and technical relations. They can be handled with the help of the theory of games. Problems of linear programming and activity analysis can provide the main basis for economic application of the theory of games.

4.10 TWO-PERSON ZERO-SUM GAME

You have already come to know that a game with two players, where a gain of one player equals the loss to the other player, is known as the **two-person zero-sum game**. In such type of game, interests of the two players are opposed so that the sum of their gains is zero. For example, if two players of chess agree that at the end of the game the loser would pay Rs. 100 to the winner of the game, then it would mean a zero sum game since the gain of one is equal to the loss of the other. On the other hand, if there are n players and sum of the game is zero then it is called **n- person zero-sum game**.

Two-person zero-sum game is also called rectangular games because their payoff matrix is in the rectangular form. A two-person zero-sum game exhibits certain characteristics as follows:

1 Only two players participate.

2. Each player has finite number of possible courses of action.

3. Each specific strategy results in a payoff.

Total payoff to the two players at the end of each play is zero.

Maximin and Minimax Strategy:

Consider a two-person zero-sum game involving the set of pure strategies $\alpha = \{A_1, A_2, A_3\}$ for player A and $\beta = \{B_1, B_2\}$ for player B and having the following payoff matrix for the player A.

		Player B		
		B_1	B_2	Row Minima r
Player A	A_1	9	2	2
	A_2	8	6	6
	A_3	6	4	4
Column Maxima c		9	6	

Suppose player A starts the game knowing fully well that whatever strategy he adopts, B will select that particular counter strategy which will minimize the payoff to A. Now if player A selects the strategy A_1, player B will reply by selecting B_2, as this corresponds to the minimum payoff to A in the first row corresponding to A_1. Similarly if A chooses the strategy A_2, he may gain 8 or 6 depending upon the strategy chosen by B. However, A can guarantee a gain of at least minimum $\{8,6\} = 6$ regardless of the strategy chosen by B. thus whatever strategy A may adopt, he can guarantee only minimum of the corresponding row payoffs. These corresponding to each $A_i \in \infty$ are indicated by forming a column vector $r = \{2, 6, 4\}$ of the row minima. Naturally A would like to maximize his gain, which is just the largest component of r. in the above example, the selection of A_2 will give the maximum (max $\{2,6,4\} = 6$) of the minimum gains to A. we can call this gain as the maximin value of the game and the corresponding strategy is called as **maximin strategy**.

On the other hand, player B wishes to minimize his loses. If he plays strategy B_1, his loss is at most max $\{9, 8, 6\} = 9$ regardless of what strategy A has followed. He can lose no more than max $\{2, 6, 4\} = 6$ if he plays B_2. These maximum loses corresponding to each $B \in \beta$ are indicated by forming a row vector $C = \{9,6\}$ of the column maxima. The smallest component of C represents the minimum possible loss to B whatever strategy he adopts. This minimum of maximum loses will be called the mimimax value of the game and the corresponding strategy is called as the **minimax strategy**.

Thus from the above example it is seen that the maximum of row minimum is equal to the minimum of the column maxima. In symbols,

$$\text{Max} \{r_i\} = 6 = \min \{c_i\}$$

Or, $\max_i [\min_j \{a_{ij}\}] = 6 = \min_j [\max_i \{a_{ij}\}]$

The selection of maximin and minimax strategies by the players A and B was based upon the so called **maximin-minimax principle**, which guarantees the best of the worst results. The corresponding pure strategies where both maximin and minimax value of the game are equal, called as the **optimum strategies**.

Saddle Point:

A saddle point of a payoff matrix is that position in the matrix where the maximum of row minima coincides with the minimum of the column maxima. The corresponding payoff at the saddle point is called the value of the game and is obviously equal to the maximin and minimax values of the game. In general, the following rules are followed for determining saddle point:

(a) Select the minimum element of each row of the payoff matrix and mark them.

(b) Select the greatest elements of each column of the payoff matrix and mark them.

(c) If there appears an element in the payoff matrix marked, the position of that element is a saddle point of the payoff matrix.

On the other hand, sometimes it is not possible to get saddle point by using pure strategy. Under such situation one has to use mixed strategies by mixing some or all of their possible courses of action to get the best possible strategy.

4.11 SADDLE POINT WITH PURE STRATEGIES

The learner has already come to know the procedure of finding the saddle point with pure strategies. Let us now take few examples to illustrate the solution of games with pure strategies.

Example1:

		Player B		
		B_1	B_2	B_3
Player A	A_1	1	3	1
	A_2	0	−4	−3
	A_3	1	5	−1

Solution: From the given pay-off matrix we have to find out minimum payoff in each row and minimum payoff for each column. The row minima vector ' r ' is obtained by writing the minimum payoff of each row. The largest component of ' r ' represents the minimum value ' $\underline{v}$ '. The column maxima vector c is obtained by writing the maximum payoff of each column. The smallest component of c represents the minimax value $\bar{v}$.

		Player B			
		B_1	B_2	B_3	Row Minima r
Player A	A_1	1	3	1	1
	A_2	0	−4	−3	−4
	A_3	1	5	−1	−1
Column Maxima c		1	5	1	

The matrix has two saddle points at positions (1,1) and (1,3). Thus the solution to the game is given by:

(a) The optimum strategy for player A is A_1.

(b) The optimum strategy for player B is either B_1 or B_3, i.e, B can use either of the two strategies.

(c) The value of the game is 1 for A and −1 for B.

Example2:

		Player B		
Player A		B_1	B_2	B_3
	A_1	6	8	6
	A_2	4	12	2

Solution:

From the given pay-off matrix we have to find out minimum payoff in each row and minimum payoff for each column. The row minima vector 'r' is obtained by writing the minimum payoff of each row. The largest component of 'r' represents the minimum value '$\underline{v}$'. The column maxima vector c is obtained by writing the maximum payoff of each column. The smallest component of c represents the minimax value $\overline{v}$.

		Player B			
Player A		B_1	B_2	B_3	Row Minima r
	A_1	6	8	6	6
	A_2	4	12	2	2
	Column Maxima c	6	12	6	

The matrix has two saddle points at positions (1,1) and (1,3), that is, saddle point exists at A_1B_1 and at A_1B_3 . Thus the solution to the game is given by:

(a) The optimum strategy for player A is A_1.

(b) The optimum strategy for player B is B_3.

(c) The value of the game is 6 for A and –6 for B.

Check Your Progress 1: Solve the following game:

		Player B		
Player A		B_1	B_2	B_3
	A_1	12	16	12
	A_2	8	24	4

Example 3:

Two competing firms want to open their next branch at one of the three cities: A, B and C, whose distance profile is given below. If both companies open their branches in the same city, they will split the business evenly. If, however, they build the branches in different cities, the company that is closer to the remaining city will get all that city's business. If all the three cities have the same amount of business of Rs. 100 lakhs, where should the firm open the branches?

Distance Profile

From A to B 25 km

From A to C 18 km

From B to C 22 km

Solution:

Let the strategies A_1, A_2, and A_3 implies that the firm I opens the branch at cities A, B and C respectively. Similarly, strategies B_1, B_2 and B_3 imply that firm II opens the branch at cities A, B and C respectively. The pay-off matrix for the firm I is given below:

		Firm II		
		B_1	B_2	B_3
Firm I	A_1	150	200	100
	A_2	100	150	100
	A_3	200	200	(150)

Saddle point occurs at the point when the firm I adopts strategy A_1 and firm II adopts strategy B_3, where,

$\max_i [\min_j \{a_{ij}\}] = 150 = \min_j [\max_i \{a_{ij}\}]$

i.e., both the firms should open their branches in city C.

The value of the game to firm I is Rs. 150 lakhs.

4.12 SADDLE POINT WITH MIXED STRATEGIES

In some game no saddle point occurs and under such circumstances it is not possible to find out its solution in terms of the maximin-minimax strategy. Games without saddle points are not strictly determined. Let us take the following game:

$$\begin{array}{lcc} & \begin{bmatrix} 6 & -2 & 3 \\ -4 & 5 & 4 \end{bmatrix} & \begin{array}{c} \text{Row Minima} \\ -2 \\ -4 \end{array} \\ \text{Column Maxima} & \begin{matrix} 6 & \;\;5 & \;\;4 \end{matrix} & \end{array}$$

In the above example, $\max . \min a_{ij} = -2 < 4 = \min . \max a_{ij}$.

The solution of such problems can be obtained by using mixed strategy. A mixed strategy refers to a combination of two or more strategies that are selected one at a time, according to pre-determined probabilities, that is, at the time of using the mixed strategy a player has to make decision to mix his choices among several alternatives in a certain ratio. For any given two-person zero-sum game where there is no saddle point, the game is solved through mixed strategies in the following manner:

Given the game,

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Let player A's strategies are represented by probabilities (p_1, p_2) and that of B's by (q_1, q_2). Now the values of p_1, p_2, q_1 and q_2 are determined by following the following formulae,

$$p_1 = \frac{a_{22} - a_{21}}{a_{11} + a_{22} - a_{21} - a_{12}}, \qquad p_2 = \frac{a_{11} - a_{12}}{a_{11} + a_{22} - a_{21} - a_{12}}$$

$$q_1 = \frac{a_{22} - a_{12}}{a_{11} + a_{22} - a_{21} - a_{12}}, \qquad q_2 = \frac{a_{11} - a_{21}}{a_{11} + a_{22} - a_{21} - a_{12}}$$

The value of the game for player A is given by

$$V = \frac{a_{11}a_{22} - a_{21}a_{12}}{a_{11} + a_{22} - a_{21} - a_{12}}$$

Example: Solve the following game:

$$\begin{bmatrix} 3 & -5 \\ -1 & 1 \end{bmatrix}$$

Solution:

By applying the formulae of mixed strategies,

$$p_1 = \frac{a_{22} - a_{21}}{a_{11} + a_{22} - a_{21} - a_{12}} = \frac{1-(-1)}{3+1-(-1)-(-5)} = \frac{2}{10}$$

$$p_2 = \frac{a_{11} - a_{12}}{a_{11} + a_{22} - a_{21} - a_{12}} = \frac{3-(-5)}{3+1-(-1)-(-5)} = \frac{8}{10}$$

$$q_1 = \frac{a_{22} - a_{12}}{a_{11} + a_{22} - a_{21} - a_{12}} = \frac{1-(-5)}{3+1-(-1)-(-5)} = \frac{6}{10}$$

$$q_2 = \frac{a_{11} - a_{21}}{a_{11} + a_{22} - a_{21} - a_{12}} = \frac{3-(-1)}{3+1-(-1)-(-5)} = \frac{4}{10}$$

The optimal strategy for the first player is $\left(\frac{2}{10}, \frac{8}{10}\right)$ and for the second player is $\left(\frac{6}{10}, \frac{4}{10}\right)$

The value of the game for player A is given by

$$V = \frac{a_{11}a_{22} - a_{21}a_{12}}{a_{11} + a_{22} - a_{21} - a_{12}} = -\frac{20}{100} = -\frac{1}{5}$$

Example: Solve the following game

	Player B	
Player A	28	0
	2	12

Solution: By applying the formulae of mixed strategies,

$$p_1 = \frac{a_{22} - a_{21}}{a_{11} + a_{22} - a_{21} - a_{12}} = \frac{12-2}{28+12-2-0} = \frac{10}{38}$$

$$p_2 = \frac{a_{11} - a_{12}}{a_{11} + a_{22} - a_{21} - a_{12}} = \frac{28-0}{28+12-2-0} = \frac{28}{38}$$

$$q_1 = \frac{a_{22} - a_{12}}{a_{11} + a_{22} - a_{21} - a_{12}} = \frac{12-0}{28+12-2-0} = \frac{12}{38}$$

$$q_2 = \frac{a_{11} - a_{21}}{a_{11} + a_{22} - a_{21} - a_{12}} = \frac{28-2}{28+12-2-0} = \frac{26}{38}$$

The optimal strategy for the first player is $\left(\frac{10}{38}, \frac{28}{38}\right)$ and for the second player is $\left(\frac{12}{38}, \frac{26}{38}\right)$.

The value of the game for player A is given by

$$V = \frac{a_{11}a_{22} - a_{21}a_{12}}{a_{11} + a_{22} - a_{21} - a_{12}} = \frac{28 \times 12 - 2 \times 0}{28 + 12 - 2 - 0} = \frac{336}{38} = 8.84$$

Check Your Progress 2

Solve the following game:

		Player B	
Player A		B_1	B_2
	A_1	8	–7
	A_2	–6	4

(**Hint:** This is a problem of mixed strategies)

4.13 DOMINANCE RULES

In a game it is possible to get a strategy available to a player preferable to some other strategy(ies). Such a strategy is said to dominate the other strategy(ies). The dominance rules are of utmost importance in simplifying the games. Let us illustrate the dominance rules with examples.

	Player B		
Player A	–6	4	5
	9	1	6
	–6	–1	1

Let us take the strategies open to player A. Here, every element of the second strategy (row) exceeds the corresponding element of the third strategy (row). The player A shall never play strategy no. 3. Thus the third strategy is dominated by the second strategy. Hence following the rule of dominance the above game is reduced to:

	Player B	
Player A	–6	4
	9	1

Similarly in the above game it is seen that the values of the third column are greater than their counterparts in the second column. Since player B would like to minimize the pay-offs to A, it would always prefer to choose second column instead of the third. Thus the third strategy is dominated by the second. The game is, thus, reduced to:

	Player B		
Player A	–6	4	5
	9	1	6

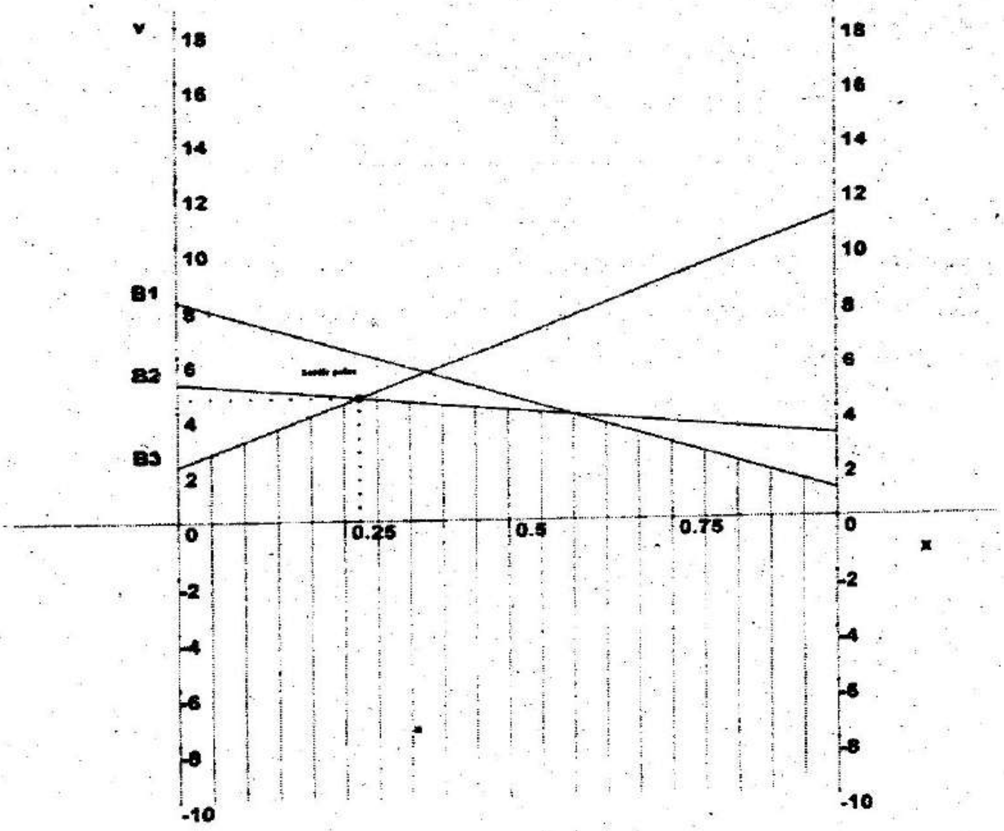
Thus, by applying the rule of dominance on a player's strategy in a game we get a reduced game.

4.14 GRAPHIC SOLUTION

A two-person zero-sum game of 2×2, $2 \times n$ or $m \times 2$ types can also be solved by graphic method. Here, the straight lines are drawn to represent respective strategies. For each value of x, the height of lines at that point denotes the pay-offs of each of B's strategies against (x, 1 − x) for A. A is concerned with his least pay-off when he plays a particular strategy and wishes to choose x so as to maximize his minimum pay-off. The value of the game is shown by the dotted vertical lines in the following graphs.

Example: Solve the following game graphically:

	Player B		
Player A	1	3	11
	8	5	2



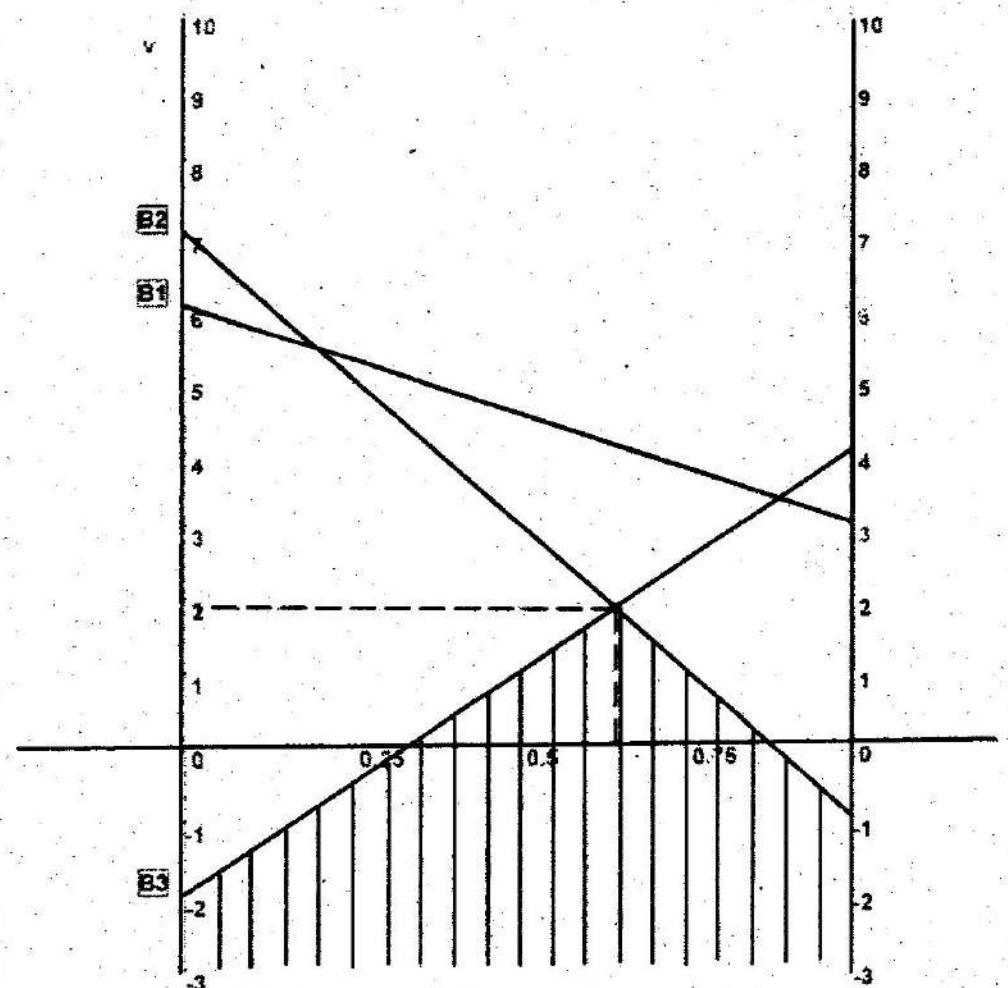
The optimal strategy for the first player is (0.27, 0.73) and for the second player is (0, 0.82, 0.18). The value of the game for player A is 4.45.

Example:

Solve the following game graphically:

	Player B		
Player A	3	−1	4
	6	7	−2

Solution:



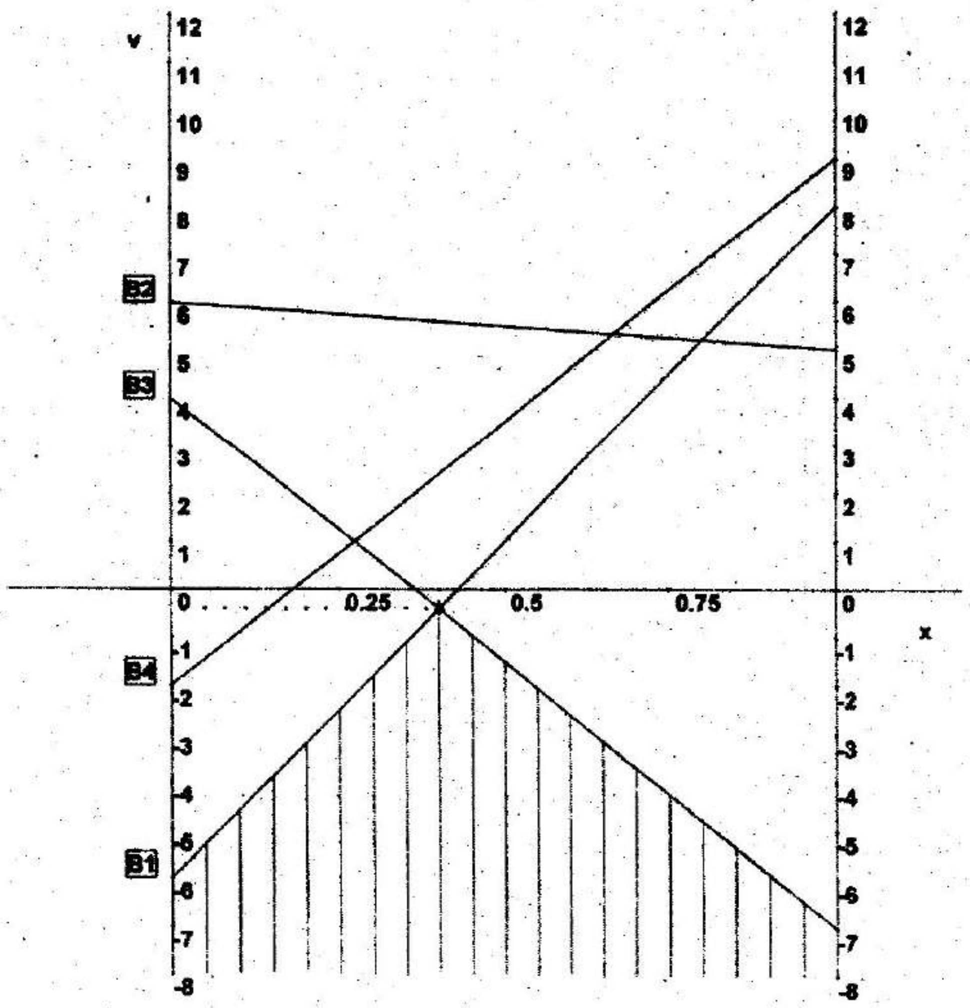
The optimal strategy for the first player is (0.64, 0.36) and for the second player is (0, 0.43, 0.57). The value of the game for player A is 1.86.

Example:

Solve the following game graphically:

	Player B			
Player A	8	5	–7	9
	–6	6	4	–2

Solution:



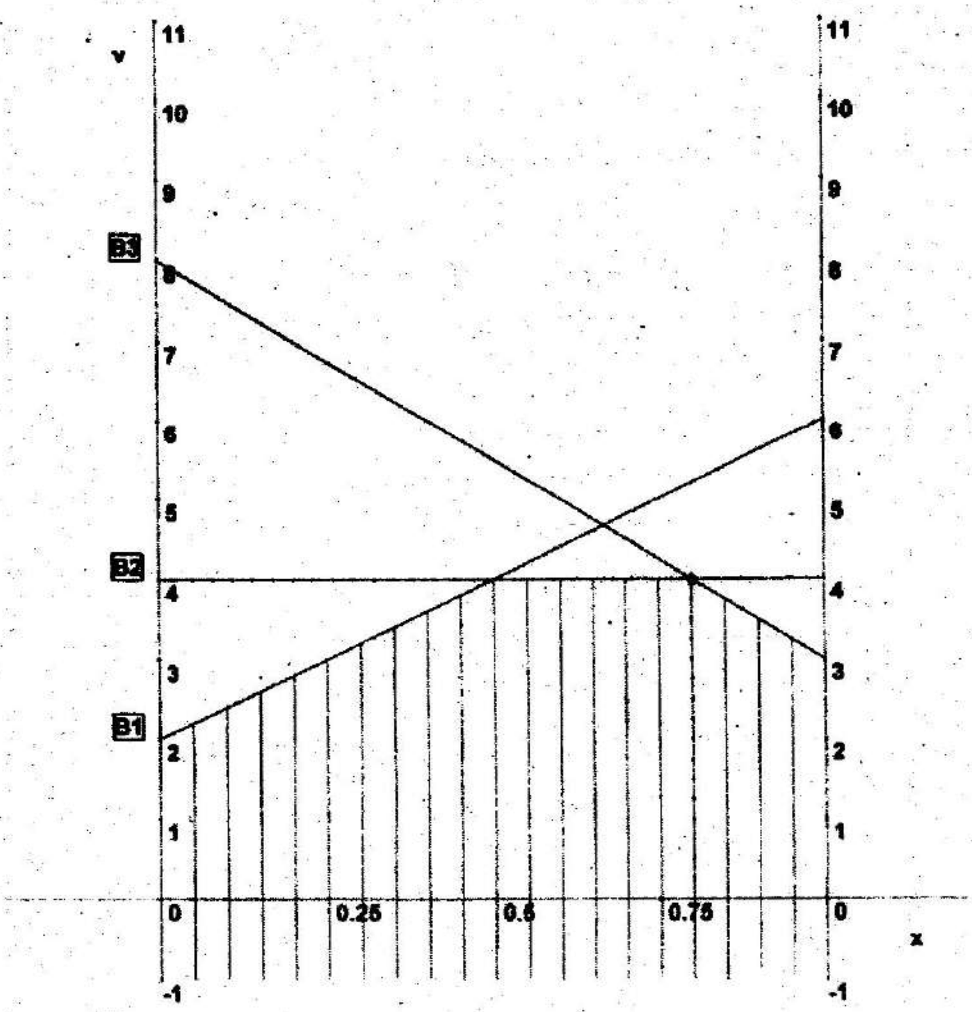
The optimal strategy for the first player is (0.40, 0.60) and for the second player is (0.44, 0, 0.56, 0). The value of the game for player A is -0.40.

Example:

Solve the following game graphically:

	Player B		
Player A	6	4	3
	2	4	8

Solution:



The optimal strategy for the first player is (0.80, 0.20) and for the second player is (0, 1, 0). The value of the game for player A is 4.

Check Your Progress 3

Solve the following game graphically:

	Player B		
Player A	2	5	6
	7	3	4

4.15 Dominant Strategy and Nash Equilibrium:

The dominant strategy is the optimal choice of the player, no mater what his/her opponent does. It is a strategy that yields higher payoffs to a player for every strategy of his/her opponent. In other words if both the firms have dominant strategy, each of them can choose their own optimal strategy regardless what their opponents are doing. On the other hand, not all games have a dominant strategy for each player. Under such situation, Nash Equilibrium exists. According to Nash, firms reach their equilibrium state when they are doing their best, given what its competitors are doing. In terms of price theory, doing their best means maximizing profits and what others are doing means what rate of output their opponents are producing or what price they are charging or what advertising expenditure they are incurring to promote the sales of their products. When each firm is doing its best, given what others are doing, no one has any incentives to change its behaviour and hence equilibrium exists. Nash Equilibrium describes a set of strategies where each player believes that it is doing the best it can, given the strategy of the other player. More specifically, the Nash Equilibrium is a situation in which each player chooses an optimal strategy, given the strategy chosen by the other player. Cournot solution is an example of Nash Equilibrium. Not all games have a Nash Equilibrium and some games have more than one. For example, if the payoffs of firm 1 and firm 2 are

	Advertisement	Do not Advertisement
Advertisement	(4, 3)	(5, 1)
Do not Advertisement	(2, 5)	(3, 2)

That is

Firm 1's payoff

	Adv.	Do not
Adv.	4	5
Do not	2	3

Firm 2's payoff

	Adv.	Do not
Adv.	3	1
Do not	5	2

It is clear that for both firms advertisement is dominant strategy. There equilibrium position will be (adv., adv.) or (4, 3). It is called dominant strategy equilibrium. But, if the firm 1's payoff changes to

	Adv.	Do not
Adv.	4	5
Do not	2	6

then for firm 2, advertisement is dominant strategy. So it always goes for advertisement without considering what firm 1 does. But firm 1 has not clear cut dominance. Therefore, it will be guided by firm 2's action.

When firm 2 advertises,

if firm 1 advertises, payoff of firm 1 = 4

if firm 1 do not advertises, payoff of firm 1 = 2

Hence, it is better for firm 1 to advertise when firm 2 advertises. The equilibrium (4,3) is called the Nash equilibrium.

4.16 Prisoner's Dilemma

Unlike zero sum games where one firm's loss is necessarily the other firm's gain and vice versa, the non zero sum or variable sum games refer to a situation where there exists a jointly preferred outcome. The existence of a jointly preferred outcome means that both players may be able to increase their payoffs through some firms of cooperation or agreement.

The variable sum games are divided into --cooperative games and non cooperative games. In a cooperative game, the players are assumed to be rational to realize that it is mutually advantageous to cooperate on any and every action which is likely to benefit at least one of the players without affecting the other adversely. But in a non cooperative game, there is no communication between the participants and there is no way to reach or enforce agreements. The most popular form of a non cooperative game is known as the Prisoner's Dilemma. It can be explained with the help of the following example :

Two criminals are arrested after committing a robbery. However, in absence of adequate evidence, their conviction depends upon the expression of one or both these criminals. If neither suspect confesses, it will be difficult to convict them and each one of them will receive only a minor sentence. On the other hand, the prosecutor promises no punishment to the one who confesses and heavy sentence (say 15 years in jail) for the one who does not confess. If both of them confess then each will receive an intermediate term punishment. Each suspect is interrogated in isolation, i.e. there is no communication between them and therefore, neither of them knows what the other is going to do. So, each suspect has the following two strategies and accordingly the following payoffs :

		Suspect 2	
		Not Confess	Confess
Suspect 1	Not Confess	No prison term for both (0, 0)	15 years' prison term for 1 and suspended sentence for 2 (15,0)
	Not Confess	Suspended sentence for 1 and 15 years' prison term for 2 (0,15)	8 years' prison term for both (8,8)

Given the lack of communication between the suspects and the uncertainty as to the loyalty of other suspect, each one of them prefers to adopt the second strategy (confess, confess). So, they get 8 years' imprisonment. Certainly they are worse off than a situation where none of them would have confessed so that they would have gone free. Due to lack of communication and trust, they are in a worse situation than they could have been. However, the decision of each suspect in favour of confession is quite rational because each person works in self-interest tries to make the 'best' of the 'worse' outcomes in an uncertain situation. Thus, situation of non zero sum games can be achieved with the help of prisoner's dilemma.

4.17 Repeated Game :

In game theory, a repeated game is an extensive form of game which consists in some number of repeatitions of some base game (called a stage game). In other words, when players interact by playing a similar stage game (such as the Prisoner's Dilemma) numerous times, the game is called a repeated game. The number of repeatitions may be finite or infinite and accordingly games are called finitely repeated and infinitely repeated games. The number of repeatitions may be known in advance, or there may be an expectation that the game will be repeated. Participants in repeated games have an incentive to choose their strategies taking into account how their actions in each play of the game will affect their reputation, that is, how other participants will expect them to behave in future rounds of the game. One striking feature of many games eg. Prisoner's Dilemma is that the Nash equilibria are non cooperative. Each player would prefer to fink than to cooperate. Repeated games can incorporate phenomena in which case cooperative behaviour can be established as a result of rational behaviour.

4.18 LET US SUM UP

In this unit you have come to know the concept game theory and the variants of game. We shall also acquaint you with the importance and limitations of game ant its application in duopoly market.

Further, you have learnt the concept of two-person zero-sum game. In zero-sum game, one player's payoff should be the negative of the payoff to the other player. You have also learnt in details about the solution procedures of two-person zero-sum game. The graphic method can also be applied to solve games in appropriate places.

4.19 KEY WORDS

Game: Game theory is a discipline concerned with the decision making in situations where two or more rational players are involved under competitive conditions and with conflicting interests.

Two-Person Zero-Sum Game: A game with two players, where a gain of one player equals the loss to the other player is known as the two-person zero-sum game.

Duopoly: It is a market situation where there are only two sellers in the market competing with each other.

Two-Person Zero-Sum Game:

A game with two players, where a gain of one player equals the loss to the other player is known as the two-person zero-sum game.

Saddle Point:

A saddle point of a payoff matrix is that position in the matrix where the maximum of row minima coincides with the minimum of the column maxima.

Value of a Game:

The corresponding payoff at the saddle point is called the value of the game and is equal to the maximin and minimax values of the game.

4.20 TERMINAL QUESTIONS

1. What is game theory? Explain its importance in economics.

2. Explain the concept of two-person zero-sum game.

3. Solve the following two-person games:

(a) $\begin{bmatrix} 15 & 0 & -2 \\ 0 & -15 & -1 \\ 1 & 2 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 4 & 0 & 1 \\ 5 & 3 & 8 \end{bmatrix}$

4. Solve the following game graphically:

	B's strategy	
A's strategy	−3	6
	8	2
	6	3

3. Explain the rules of dominance in the theory of game.

4.21 ADDITIONAL READINGS

Koutsoyiannis A (1991): Modern Microeconomics, ELBS-Macmillan, Hampshire.

Taha Hamdy A (2003): Operations Research, Pearson education, New Delhi.

Vohra N D (2001): Quantitative Techniques in Management, Tata McGraw Hill, New Delhi.

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